CS 598 YYZ
Advanced Computer Graphics

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Generalized Coordinates and Inverse Kinematics

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Generalized Coordinates

We have a serires of n point masses ri, where i = 1 to n and a series of k constraints, gj(r1...rn) where j = 1..k.
This means we have 3n-k degrees of freedom.
These degrees of freedom are generalized coordinates, ql, where l = 1 ... 3n-k.
We can recover the original coordinates as a function of the generalized coordinates: ri = fi(q1,...,qn)

Forward Kinematics

Forward kinematics produces a mapping between generalized coordinates and actual coordinates.

An articulated body is typically
-organized into hierarchical structure
-transformation at each node
-translation to local coords at each joint
-each segment is a leaf node

needs a matrix stack
-global matrix at root
-when descend, multiply current transformation matrix by the one at the node, place on the stack
-when ascend, pop matrix off of the stack
-provides a mapping of joint angles to world coordinates

The Lagrangian

The Lagrangian

- is a function similar to energy of system:

$$L(q,\dot{q}) = T(q,\dot{q}) - V(q)$$

where L is the Lagrangian, T is kinetic energy, and V is potential energy

Lagrage's Equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = \gamma_j, i = 1, \ldots n$$

-The right hand side is a generalized force.

-Particles have internal and external forces.
The generalized force is external.
(Lagragian already takes care of internal forces)

-Replace T and V in the Lagrangian with specific formula and solve

Generalized Force

Suppose there is a force Fi on ri
-force does work, so virtual displacement on ri is:

$$\sigma r_i$$

-total work done by forces is:

$$\sum_i F_i \sigma r_i$$

where σr_i is the differential of coordinates, so we can use the partial derivatives
-so,

$$F_i \sigma r_i = F_i \sum_j \frac{\partial r_i}{\partial q_j} \sigma q_j$$

Generalized Force (continued)

We want to consider total work by all forces:

$$\sum_i F_i \sigma r_i = \sum_i F_i \sum \frac{\partial r_i}{\partial q_j} \sigma_j$$

can reverse order of the summations:

$$\sum_j \sum_i F_i \frac{\partial r_i}{\partial q_j} \sigma q_j$$

The result from each inner summation represents a generalized force. The whole summation together is virtual work.

Example

Consider a pendulum. The string is of length l, the mass is m, the gravitational constant is g, the angle from the vertical is theta, the angle around the vertical is phi. We use the 2 angles as the generalized coordinates, but the pendulum has 3 DOF, so we need to map 2 DOF to 3 DOF.

-The position of the pendulum is :

$$r(\Theta, \Phi) = \begin{pmatrix} l sin\Theta cos\Phi \\ l sin\Theta sin\Phi \\ -l cos\Theta \end{pmatrix}$$

-The kinetic energy is: $(1/2)m \left\| \dot{r} \right\|^2 = .5mL^2(\dot{\Theta}^2 + (1 - cos^2|\dot{\Theta}^2)$

-Potential energy is $\mathrm{V = mgLcos\Theta}$

-The external force is 0.

Example continued

-Now we get the Lagrangian for theta and phi.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\Theta}} = mL^2\ddot{\Theta}$$
$$\frac{\partial L}{\partial \Theta} = mL^2(sin\Theta)(cos\Theta)\Theta * \Phi^2 - mgL * sin\Theta$$

-Subtract the above two equations to get the Lagrangian for theta.

-For phi:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\Phi}} = mL^2(sin^2\Theta)\ddot{\Phi} + 2mL^2(sin\Theta)(cos\Theta)\dot{\Theta}\dot{\Phi}$$
$$\frac{\partial L}{\partial \Theta} = 0$$

-Again, combine these two to get the Lagrangian for phi

Example continued

-Solve for the Diff Eq, whose final form is:

$$\begin{pmatrix} mL^2 & 0 \\ 0 & mL^2 sin^2\Theta \end{pmatrix} * \begin{pmatrix} \Theta^{\cdot\cdot} \\ \Phi^{\cdot\cdot} \end{pmatrix} + \begin{pmatrix} -mL^2(sin\Theta)(cos\Theta)\Phi^{\cdot 2} \\ 2mL^2(sin\Theta)(cos\Theta)\Theta^{\cdot}\Phi^{\cdot} \end{pmatrix} + \begin{pmatrix} mgL(sin\Theta) \\ 0 \end{pmatrix} = 0$$

-Get 2 curves as a function of time
-Map to coords for animation

Inverse Kinematics

Consider 2 bars joined together. We have two angles, between the first bar and what it is connected to, and the first bar and second bar.

-End effector is what we place, the end of the second bar, in this case.

-Given 2 angles, what is the 3D location of the endpoint is forward kinematics

-But the inverse is if we have a 3D location for the end effector, what are the angles?

Methods

-Closed-form

-Differential approach

-Optimization