

Algorithms for Column-Sparse Packing IPs

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- Packing IP

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 - UFP with bounded-length paths (e.g., expanders)

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 - $O(k^2)$ approx for k -sparse PIPs (based on Pritchard's ideas)

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- Can order the variables arbitrarily and use a similar analysis

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$$\begin{aligned} \Pr[i \text{ prevents } j \text{ from being acc}] &\leq \Pr[\exists \ell \text{ s.t. } a_{i\ell} \neq 0, y_\ell = 1] \\ &\leq \sum_{\ell: a_{i\ell} \neq 0} \Pr[y_\ell = 1] \\ &= \sum_{\ell: a_{i\ell} \neq 0} \theta x_\ell \\ &\leq 2\theta \sum_{\ell} a_{i\ell} x_\ell \\ &\leq 2\theta \\ &= 1/2k \end{aligned}$$

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If j was integral when i was removed, all of the other vars (if any) participating in i were integral when i was removed $\Rightarrow i$ is satisfied.

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Theorem. G is $(2k^2 + 1)$ -colorable and thus it has an independent set of weight at least $\text{OPT}_f / (2k^2 + 1)$.

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Corollary. $O(k^2)$ approximation.

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Remark. Bansal *et al.* recently improved the approximation ratio to $O(k)$, matching the lower bound.

THANK YOU

Questions?