

Prize-collecting Steiner Problems on Planar Graphs

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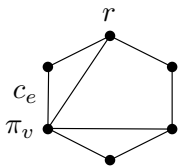
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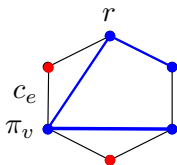
PC Steiner Tree and Forest

(Rooted) Prize-collecting Steiner tree



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$$\text{Length}(T) = \sum_{e \in T} c_e$$

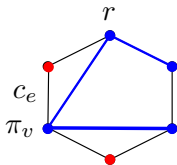
$$\text{Penalty}(T) = \sum_{v \notin T} \pi_v$$

$$\text{Cost}(T) = \text{Length}(T) + \text{Penalty}(T)$$

Generalizes Steiner tree

PC Steiner Tree and Forest

(Rooted) Prize-collecting Steiner tree



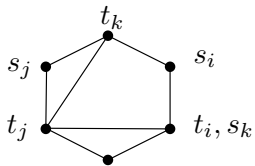
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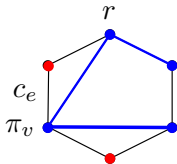
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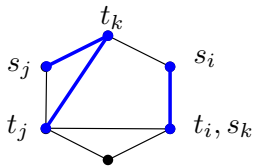
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Generalizes Steiner tree

Prize-collecting Steiner forest



$$\text{Length}(F) = \sum_{e \in F} c_e$$

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$$\text{Cost}(F) = \text{Length}(F) + \text{Penalty}(F)$$

Generalizes Steiner forest

Motivation

- Basic network design problems
- Building blocks for k -MST, Orienteering
- Practical applications

Previous Work: Highlights

Prize collecting Steiner tree

- 1.967-approximation (Archer *et al.* 2009)
- 2-approximation (Goemans-Williamson 1995)

Steiner tree in planar graphs

- PTAS (Borradaile-Klein-Mathieu 2009)

Prize collecting Steiner forest

- 2.54-approximation (Jain-Hajiaghayi 2006)
- 2.54-approximation for **submodular** PCSF (Sharma-Swamy-Williamson 2007)

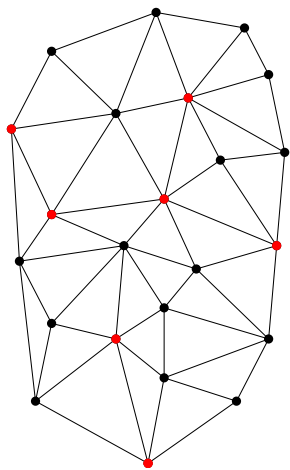
Steiner forest in planar graphs

- PTAS (Bateni-Hajiaghayi-Marx 2010)

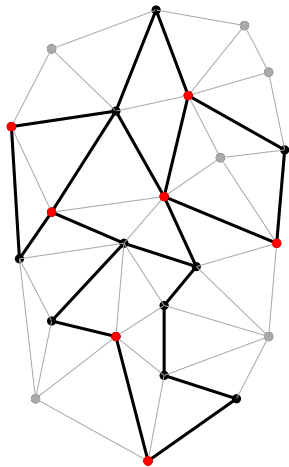
Our Results

- PTAS for PC Steiner Tree, PC Steiner TSP, PC Stroll
- Reduction to bounded tw for Submodular PC Steiner Forest
- APX-hardness for PC Steiner Forest in series-parallel graphs

Steiner Tree in Planar Graphs (Borradaile-Klein-Mathieu 2007)



Steiner Tree in Planar Graphs (Borradaile-Klein-Mathieu 2007)

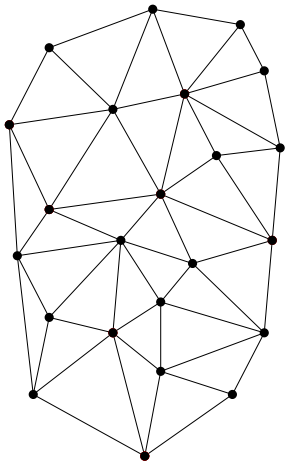


Find a special subgraph spanning the terminals

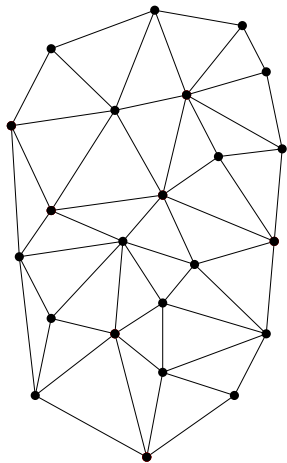
$$\text{Length}(H) \leq f(\epsilon)\text{OPT}, \text{OPT}(H) \leq (1 + \epsilon)\text{OPT}$$

Reduce to bounded treewidth
(Baker, Borradaile *et al.*, Demaine *et al.*)

Capturing Important Vertices



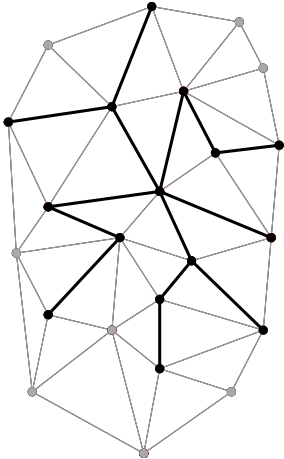
Capturing Important Vertices



Find a tree T

$$\text{Cost}(T) \leq f(\epsilon)\text{OPT} \quad \pi(T^* - T) \leq \epsilon\text{OPT}$$

Capturing Important Vertices



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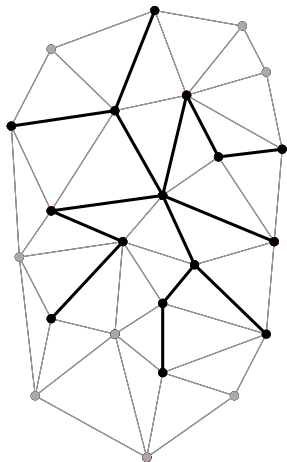
$$\text{Cost}(T) \leq f(\epsilon)\text{OPT} \quad \pi(T^* - T) \leq \epsilon\text{OPT}$$

Increase penalties

$$\pi'_v = \frac{1}{\epsilon} \cdot \pi_v$$

Run Goemans-Williamson algorithm

Capturing Important Vertices



Find a tree T

$$\text{Cost}(T) \leq f(\epsilon)\text{OPT} \quad \pi(T^* - T) \leq \epsilon\text{OPT}$$

Increase penalties

$$\pi'_v = \frac{1}{\epsilon} \cdot \pi_v$$

Run Goemans-Williamson algorithm

$$\text{Length}(T) + \pi'(V - T) \leq 2\text{OPT}' \leq \frac{2}{\epsilon}\text{OPT}$$

$$\pi'(T^* - T) \leq \text{OPT}$$

General reduction

Find (Structure step)

- partition $\{D_0, D_1, \dots, D_k\}$ of demands
- trees T_i satisfying D_i for $i > 0$

such that

- 1 $\pi(D_0)$ is small
- 2 total cost of T_i 's is $O(\text{OPT})$
- 3 solving D_i 's independently is fine

How?

- submodular primal-dual growth (scaled penalties)
ensure that necessary demands are satisfiable
- contract each comp & assign potential
- PC-Clustering of [BHM'10]
ensure that necessary connections exist

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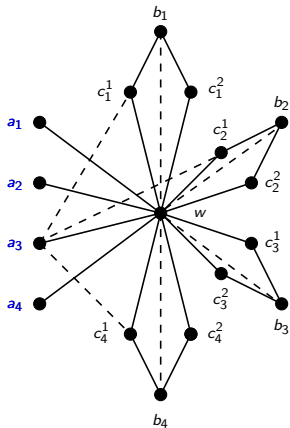
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Hardness

- PC-SF is APX-hard even on series-parallel graphs
- Separation for a natural problem between prize-collecting and normal version
- Hardness comes from interaction of demands, not from different comps: solution has only one comp
- Can be extended to Euclidean metrics
- Reduction from 3-regular vertex cover

Hardness proof



VC	PC-SF
N verts	a_i
E edges	b_j, c_j^1, c_j^2
VC	$2N + 2E + VC$

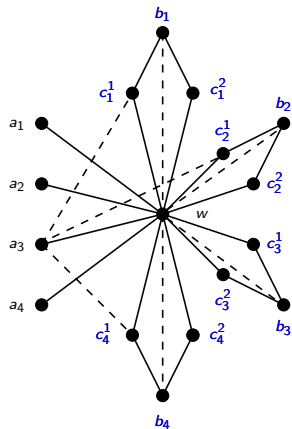
$$\text{cost}(a_i, w) = 2$$

cost of other edges = 1

$$\text{demand}(w, b_j) = 3$$

demand $(a_i, c_j^i) = 1$ if i is the endpoint

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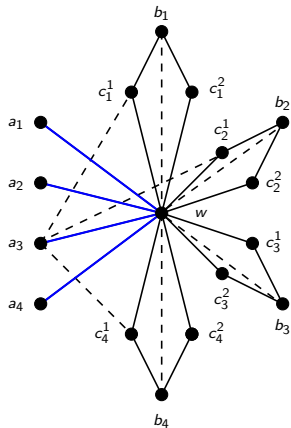
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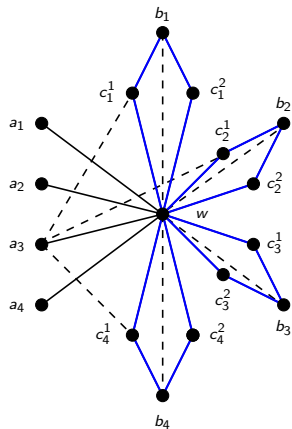
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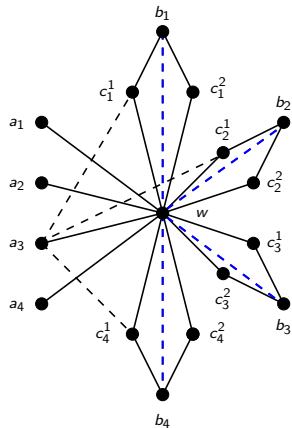
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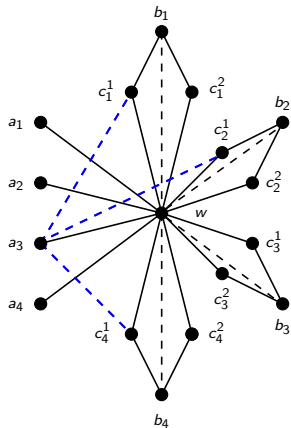
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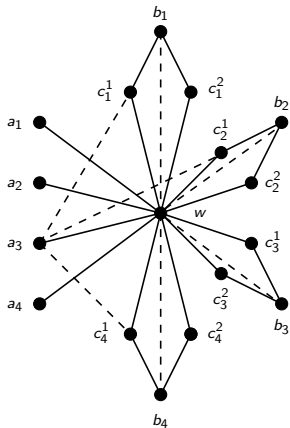
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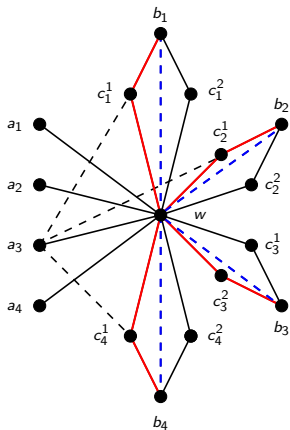
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Part 1: PCSF $\leq 2N + 2E + VC$

- add edge (a_i, w) iff $i \notin VC$
- add path via an uncovered c_j^i

Part 2: PCSF $\geq 2N + 2E + VC$

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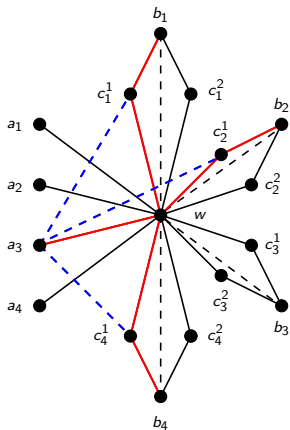
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Part 1: $\text{PCSF} \leq 2N + 2E + VC$

Part 2: $\text{PCSF} \geq 2N + 2E + VC$

- all b_j demands satisfied
- only one of c_j^1, c_j^2 has edges
- a_i has an edge iff all 3 demands satisfied
- $\{i : a_i \notin \text{PCSF}\}$ is a VC soln
- $\text{PCSF} \geq 2(N - VC) + 2E + 3VC$

Hardness proof



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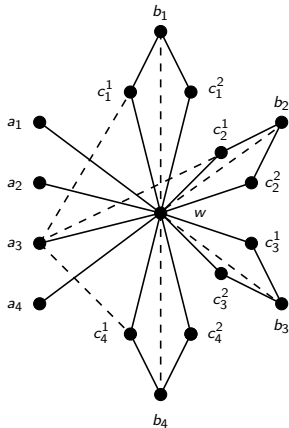
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Remarks and Open Problems

Reductions extend to bounded genus graphs

Open problems

- PTAS for planar k -MST?
- PTAS for planar Orienteering?

Thank You
Questions?