

Minimum Cost Submodular Allocation

Chandra Chekuri and [Alina Ene](#)

Theory Seminar 2011

Submodular Allocation

- Finite ground set V
- k labels
- k submodular functions f_1, \dots, f_k
- Assign a label to each element of V
- Minimize $\sum_{i=1}^k f_i(C_i)$

Monotone MCSA

- Captures Set Cover [Svitkina-Tardos 04]
- Greedy H_n approx [Svitkina-Tardos 04]
- Load balancing
- Facility location with submodular costs

Non-monotone MCSA

- Probably hard in general?
- “Uniform” non-monotone MCSA
- k terminals s_1, \dots, s_k
- $f_i(S) = f(S + s_i)$

Multiway Partition

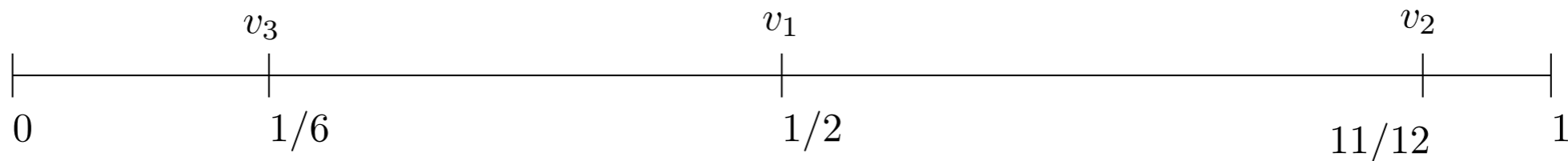
- $f : 2^V \rightarrow \mathbb{R}_+$
- k terminals s_1, \dots, s_k
- Partition V into sets C_1, \dots, C_k
- $s_i \in C_i$
- Minimize $\sum_{i=1}^k f(C_i)$

Submodular Labeling

- Partition function $f : 2^V \rightarrow \mathbb{R}_+$
- f is special (e.g., hypergraph cut, symmetric)
- Assignment costs $g_i : 2^V \rightarrow \mathbb{R}_+$
- g_i is monotone
- Partition V into sets C_1, \dots, C_k
- Minimize $\sum_{i=1}^k g_i(C_i) + \sum_{i=1}^k f(C_i)$

Lovasz Extension

$$f : \{0, 1\}^n \rightarrow \mathbf{R} \Rightarrow \hat{f} : [0, 1]^n \rightarrow \mathbf{R} \quad \hat{f}(x) = \mathbf{E}_{\theta \in [0, 1]} f(x^\theta) = \int_0^1 f(x^\theta) d\theta$$



$$\hat{f}((1/2, 11/12, 1/6)) = \left(\frac{1}{6} - 0\right) f(\{v_1, v_2, v_3\}) + \left(\frac{1}{2} - \frac{1}{6}\right) f(\{v_1, v_2\}) + \left(\frac{11}{12} - \frac{1}{2}\right) f(\{v_2\}) + \left(1 - \frac{11}{12}\right) f(\emptyset)$$

Relaxation

Lóvasz-extension-based Convex Relaxation

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^k \hat{f}_i((x(v_1, i), \dots, x(v_n, i))) \\ \text{subject to} & \sum_{i=1}^k x(v, i) = 1 \quad \forall v \\ & x(v, i) \geq 0 \quad \forall v, i \end{array}$$

\hat{f} is convex iff f is submodular [Lovasz83]

For several f , LE-Rel is an LP (e.g., cut fn)

LE-Rel Integrality Gap

- H_n for monotone MCSA (tight)
(matches [Svitkina-Tardos 04])
- 1.5 for Graph Multiway Cut
(matches [Calinescu-Karloff-Rabani 98])
- $\min(2, H_\Delta)$ for Hypergraph Multiway Cut
- 1.5 for Symmetric Hypergraph Multiway Cut
- 2 for Symmetric Multiway Partition
(matches [Queyranne 98, Zhao-Nagamochi-Ibaraki 05])

LE-Rel Gap for Labeling

- 2 for graph cut + modular costs
(matches [Kleinberg-Tardos 02, Chekuri et al 02])
- Δ for hypergraph cut + modular costs (tight)
- $O(\ln n)$ for graph cut + submod. costs
- $O(\ln n + \Delta)$ for hyper cut + submod. costs
- $O(\ln n)$ for symmetric f + submod. costs

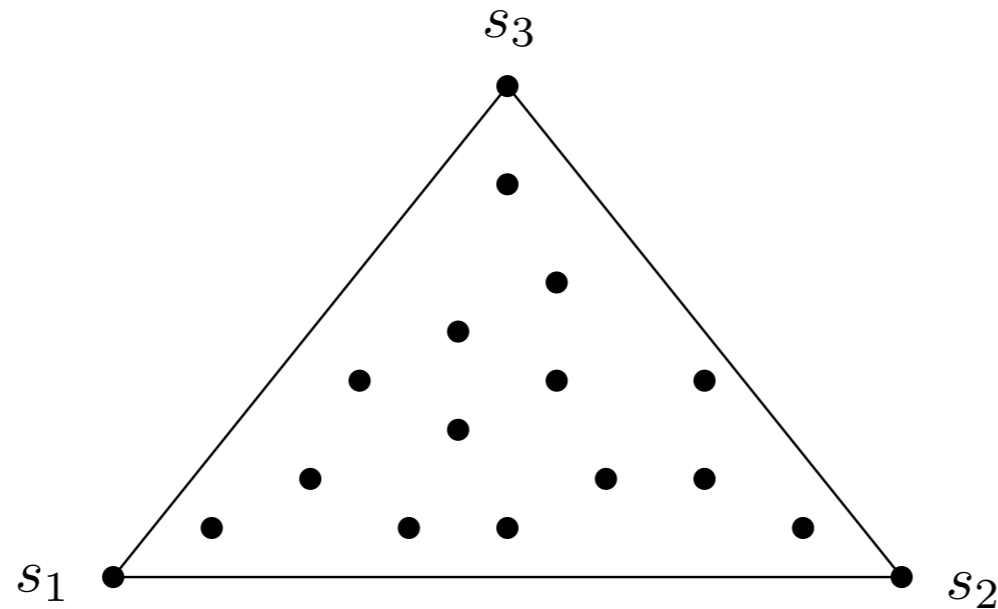
New Results

- 1.5 for Symmetric Hypergraph Multiway Cut
- 1.833 for Hypergraph Multiway Cut if $\Delta = 3$
- 1.833 for Node Multiway Cut if $\Delta = 3$
- Δ for hypergraph cut + modular costs
- $O(\ln n + \Delta)$ for hyper cut + submod. costs
- $O(\ln n)$ for symmetric f + submod. costs

Multiway Cut

Embed vertices on $(k - 1)$ -simplex

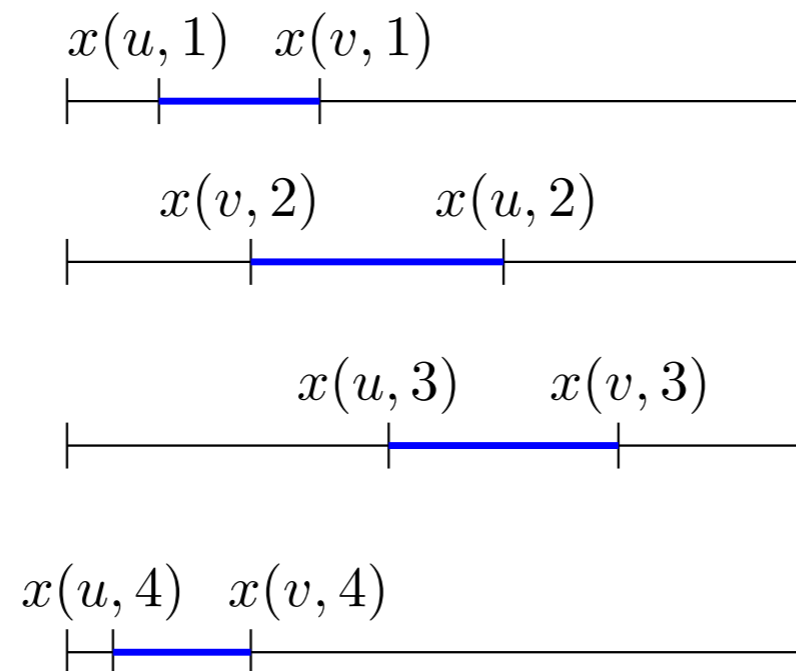
Pin terminals to vertices of the simplex



Multiway Cut

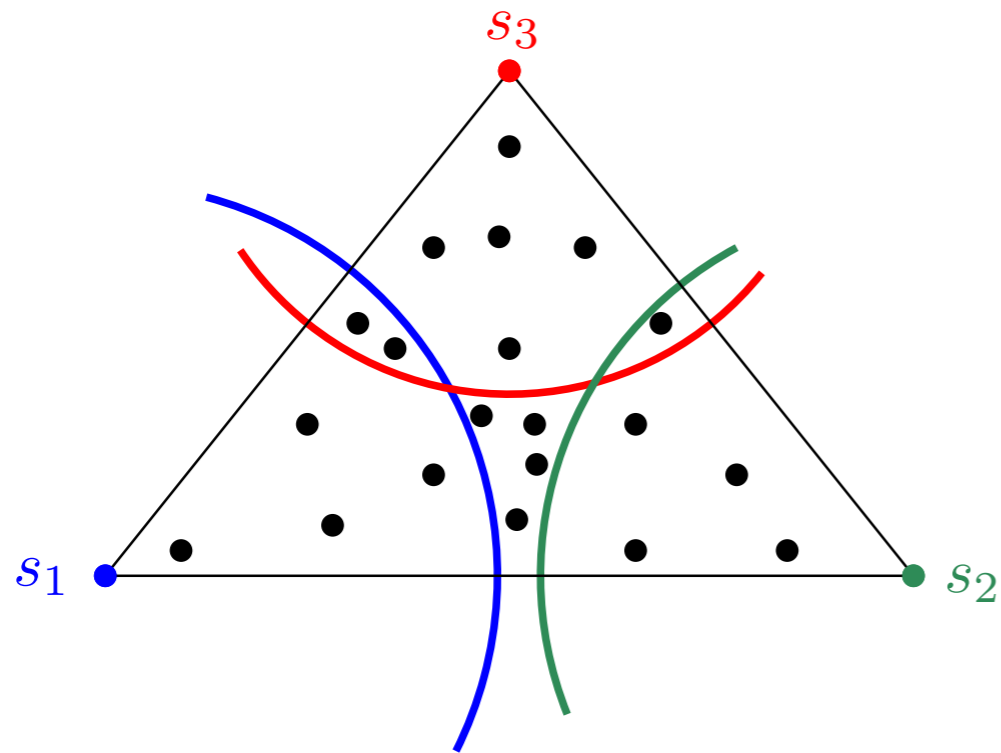
What is the objective function of LE-Rel?

$$\sum_{i=1}^k \hat{f}(\mathbf{x}_i) = \sum_{i=1}^k \mathbf{E}_{\theta \in [0,1]} [f(\mathbf{x}_i^\theta)] = \sum_{uv \in E} \sum_{i=1}^k |x(u, i) - x(v, i)| w(uv)$$

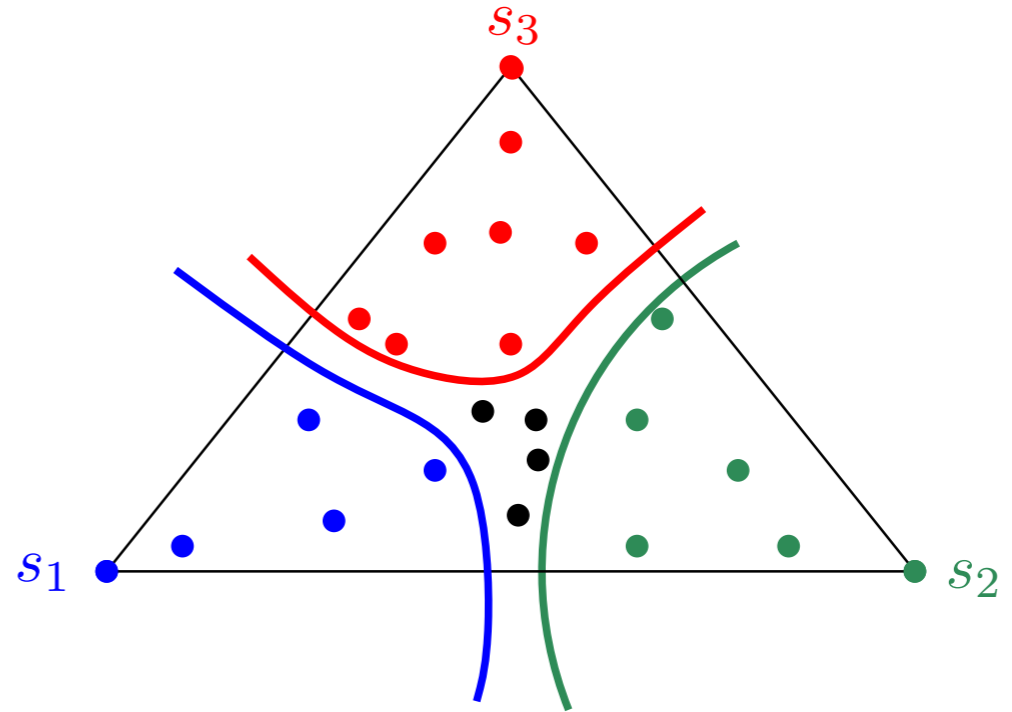


Rounding

$\theta \in_R [0, 1]$

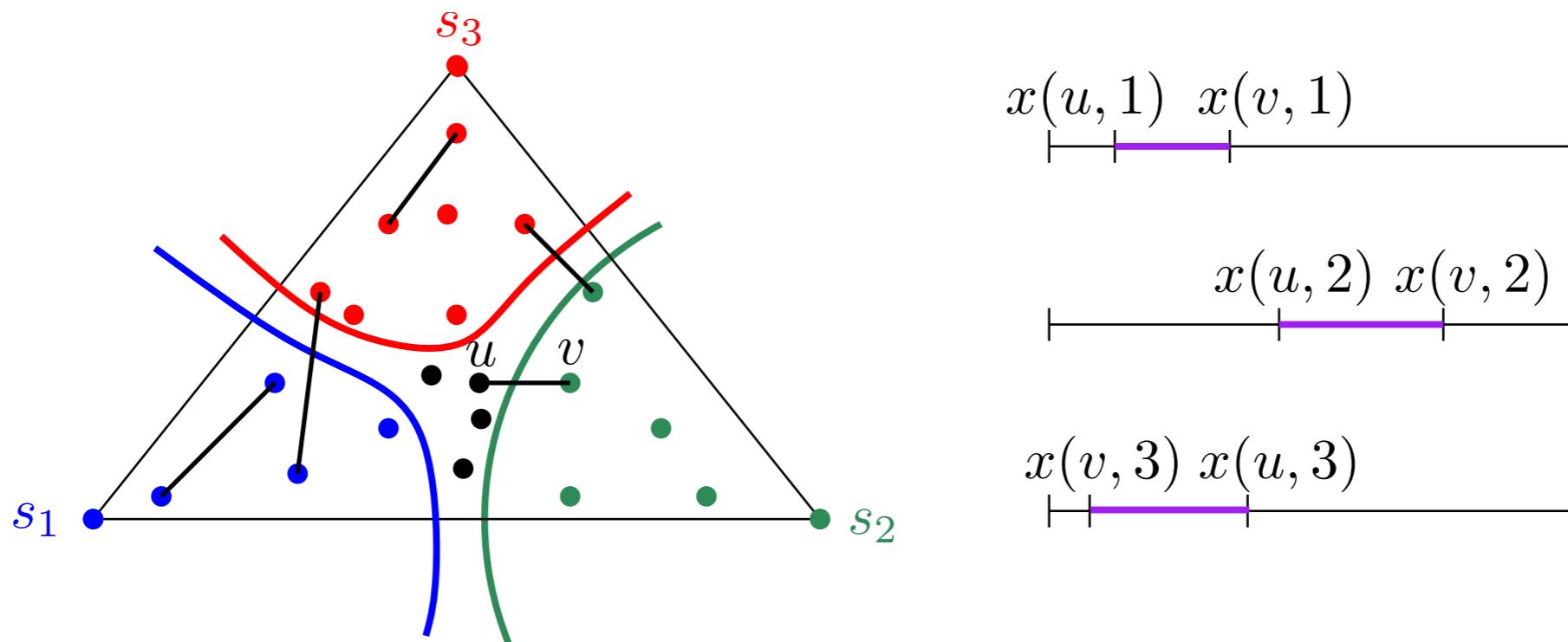


Uncrossing
(posi-modularity)



Analysis

- Uncrossing can only decrease the cost
- Cost of unassigned vertices is at most **half**



- Generalizes to hypergraphs

Conclusion

- Unified framework
- Capture new problems
- Use known roundings for new problems
- Unified rounding algorithm? (Bad gap!)
- Hard to approximate (to $\text{poly}(n)$ factors)?

Thank You!

Questions?