Solving the Minimum Biclique Cover Problem in Practice

Rob Schreiber
HP Labs

Nikola Milosavljević
Stanford University

Alina Ene
Princeton University

HP Labs 2007

The Minimum Biclique Cover Problem

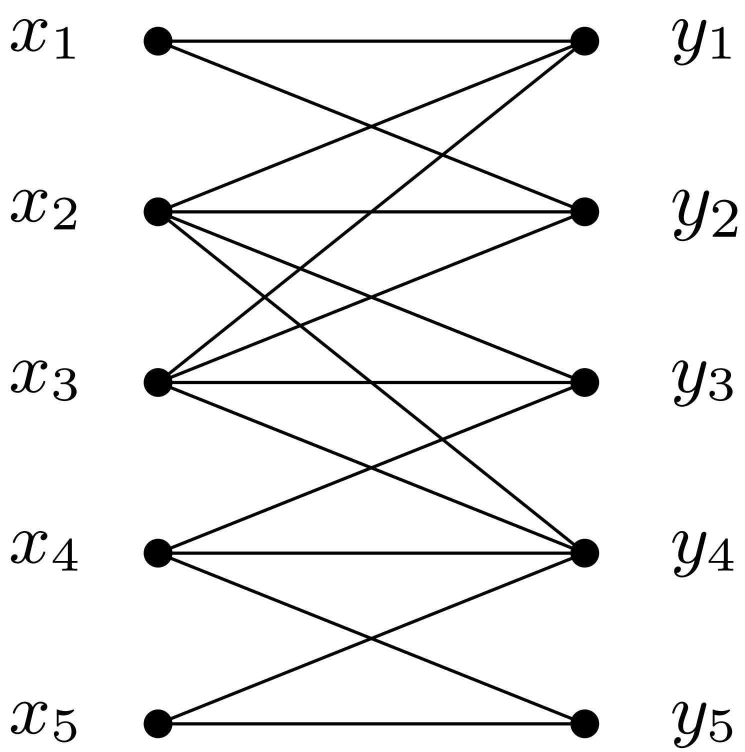
Input: A bipartite graph $G = (L, R, E)$

Problem: Find the minimum number k of bicliques such that each edge is contained in at least one biclique

The Minimum Biclique Cover Problem

Input: A bipartite graph $G = (L, R, E)$

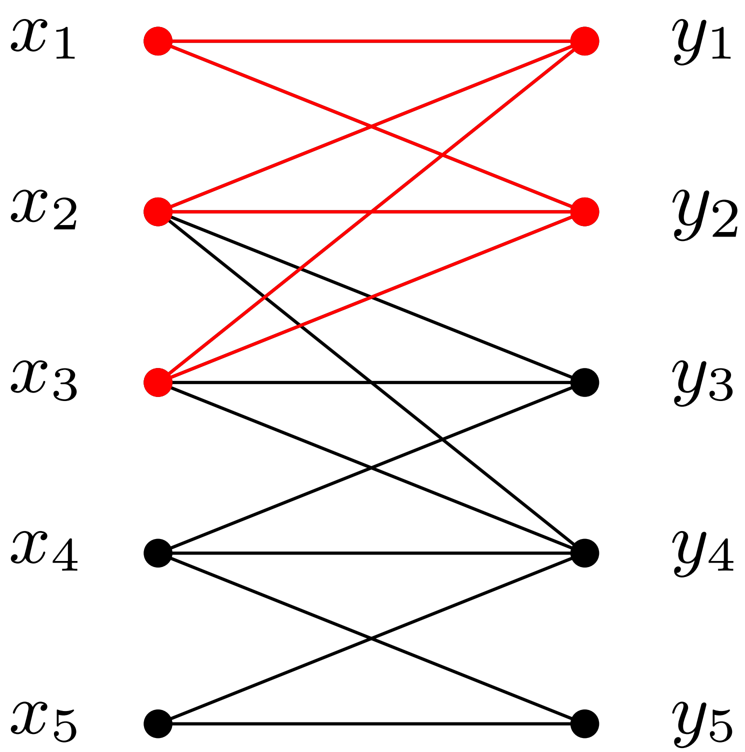
Problem: Find the minimum number k of bicliques such that each edge is contained in at least one biclique



The Minimum Biclique Cover Problem

Input: A bipartite graph $G = (L, R, E)$

Problem: Find the minimum number k of bicliques such that each edge is contained in at least one biclique

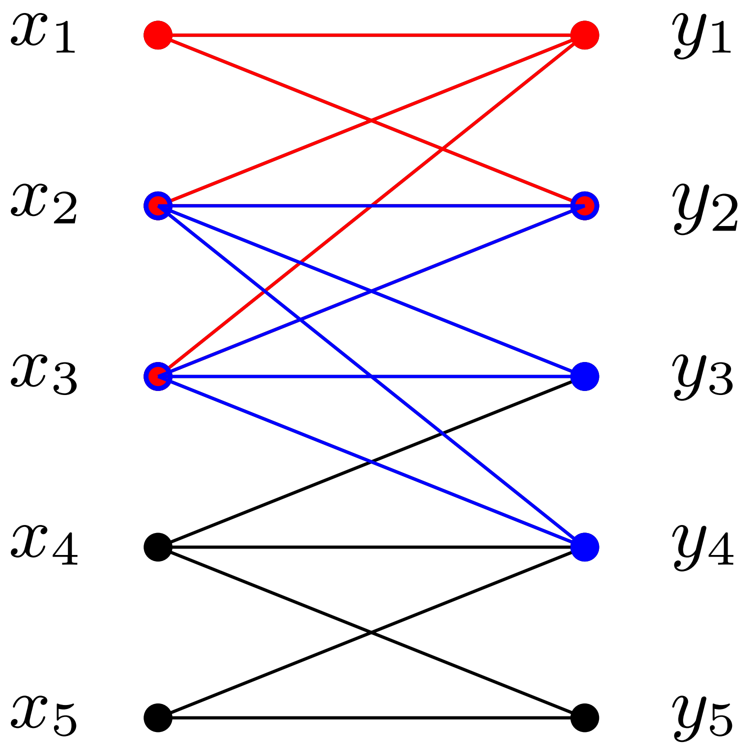


$(\{x_1, x_2, x_3\}, \{y_1, y_2\})$

The Minimum Biclique Cover Problem

Input: A bipartite graph $G = (L, R, E)$

Problem: Find the minimum number k of bicliques such that each edge is contained in at least one biclique



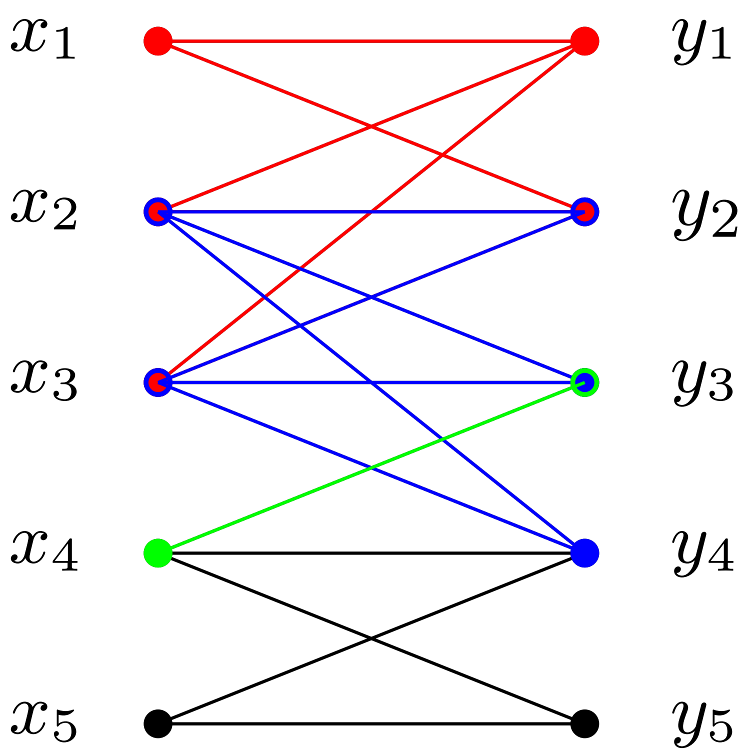
$(\{x_1, x_2, x_3\}, \{y_1, y_2\})$

$(\{x_2, x_3\}, \{y_2, y_3, y_4\})$

The Minimum Biclique Cover Problem

Input: A bipartite graph $G = (L, R, E)$

Problem: Find the minimum number k of bicliques such that each edge is contained in at least one biclique



$(\{x_1, x_2, x_3\}, \{y_1, y_2\})$

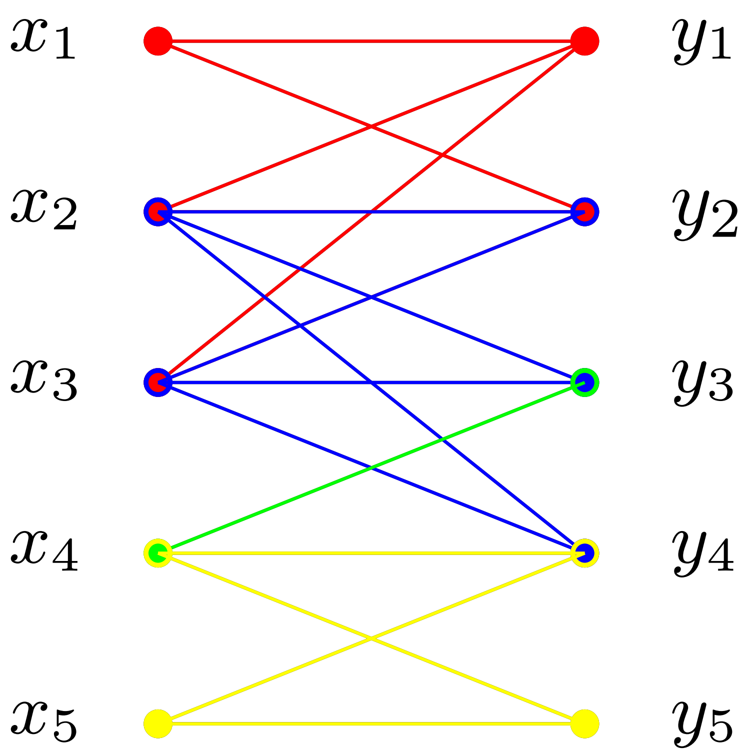
$(\{x_2, x_3\}, \{y_2, y_3, y_4\})$

$(\{x_4\}, \{y_3\})$

The Minimum Biclique Cover Problem

Input: A bipartite graph $G = (L, R, E)$

Problem: Find the minimum number k of bicliques such that each edge is contained in at least one biclique



$(\{x_1, x_2, x_3\}, \{y_1, y_2\})$

$(\{x_2, x_3\}, \{y_2, y_3, y_4\})$

$(\{x_4\}, \{y_3\})$

$(\{x_4, x_5\}, \{y_4, y_5\})$

The Minimum Biclique Cover Problem

Applications in graph drawing and clustering, biology, artificial intelligence, etc.

NP-hard [Orlin 1977].

NP-hard to approximate within n^{δ} , for some constant $\delta > 0$ [Lund and Yannakakis 1994].

Reduction Rules for **MBC**

Definition. A reduction rule replaces an **MBC** instance by a simpler instance, such that the solution to the original instance can be reconstructed from the solution of the simpler instance.

Biclique Cover (**BC**)
Input: bipartite graph G, integer k
Solution: a collection $\mathcal{B}$, $|\mathcal{B}| \leq k$, of bicliques of G such that each edge $e \in E(G)$ is contained in at least one biclique

Extended Biclique Cover (**EBC**)
Input: bipartite graph G, list of edges L, integer k
Solution: a collection $\mathcal{B}, |\mathcal{B}| \leq k$, of bicliques of G such that each edge $e \in E(G) \backslash L$ is contained in at least one biclique

Definition. A reduction rule maps an instance $\langle G, L, k \rangle$ of **EBC** to an instance $\langle G', L', k' \rangle$ such that $\langle G, L, k \rangle$ has a solution if and only if $\langle G', L', k' \rangle$ has a solution.

Irrelevant Vertex Rule

Let $\langle G, L, k \rangle$ be an instance of **EBC**.

Definition. An irrelevant vertex is incident to only edges in L.

Irrelevant Vertex Rule. Remove any irrelevant vertex from G.

Lemma. The Irrelevant Vertex Rule is safe.

Irrelevant Vertex Rule

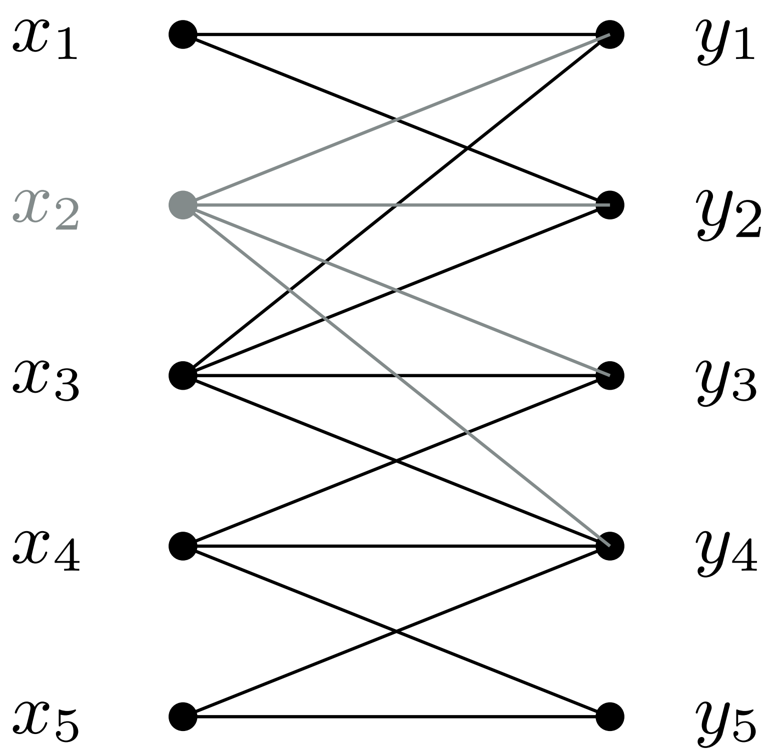
Let $\langle G, L, k \rangle$ be an instance of **EBC**.

Definition. An irrelevant vertex is incident to only edges in L.

Irrelevant Vertex Rule. Remove any irrelevant vertex from G.

Lemma. The Irrelevant Vertex Rule is safe.

$$\langle G, L, k \rangle \to \langle G \backslash \{x_2\}, L \cap E(G \backslash \{x_2\}), k \rangle$$



Irrelevant Vertex Rule

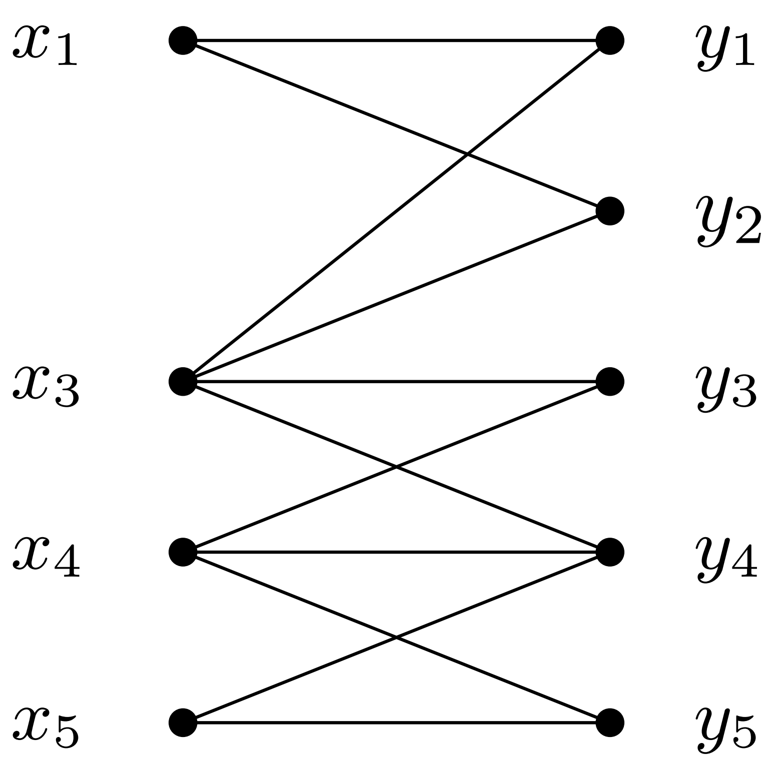
Let $\langle G, L, k\rangle$ be an instance of **EBC**.

Definition. An irrelevant vertex is incident to only edges in L.

Irrelevant Vertex Rule. Remove any irrelevant vertex from G.

Lemma. The Irrelevant Vertex Rule is safe.

$$\langle G, L, k\rangle \rightarrow \langle G\backslash\{x_2\}, L \cap E(G\backslash\{x_2\}), k\rangle$$



Irrelevant Vertex Rule

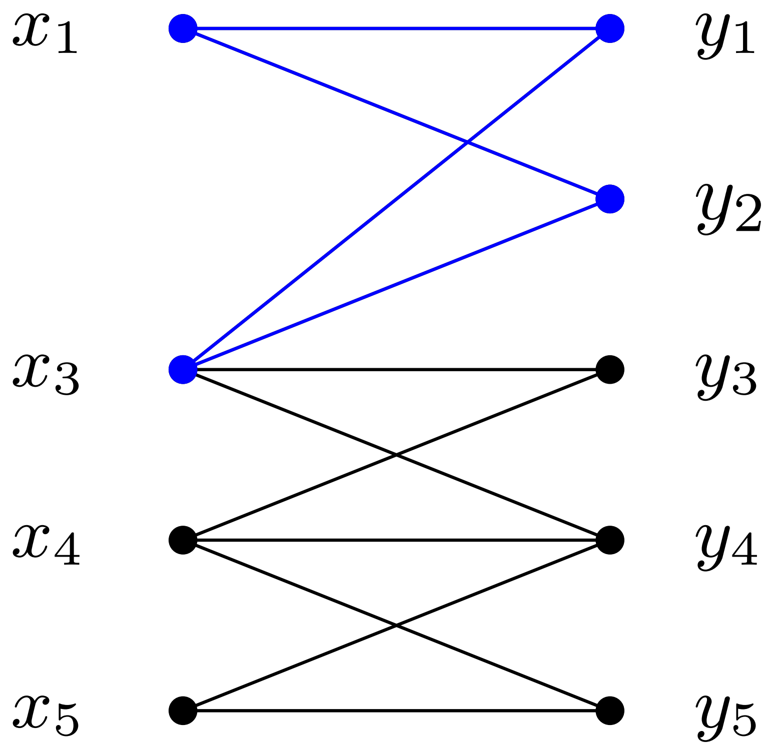
Let $\langle G, L, k \rangle$ be an instance of **EBC**.

Definition. An irrelevant vertex is incident to only edges in L.

Irrelevant Vertex Rule. Remove any irrelevant vertex from G.

Lemma. The Irrelevant Vertex Rule is safe.

$$\langle G, L, k \rangle \rightarrow \langle G \backslash \{x_2\}, L \cap E(G \backslash \{x_2\}), k \rangle$$



$(\{x_1, x_3\}, \{y_1, y_2\})$

Irrelevant Vertex Rule

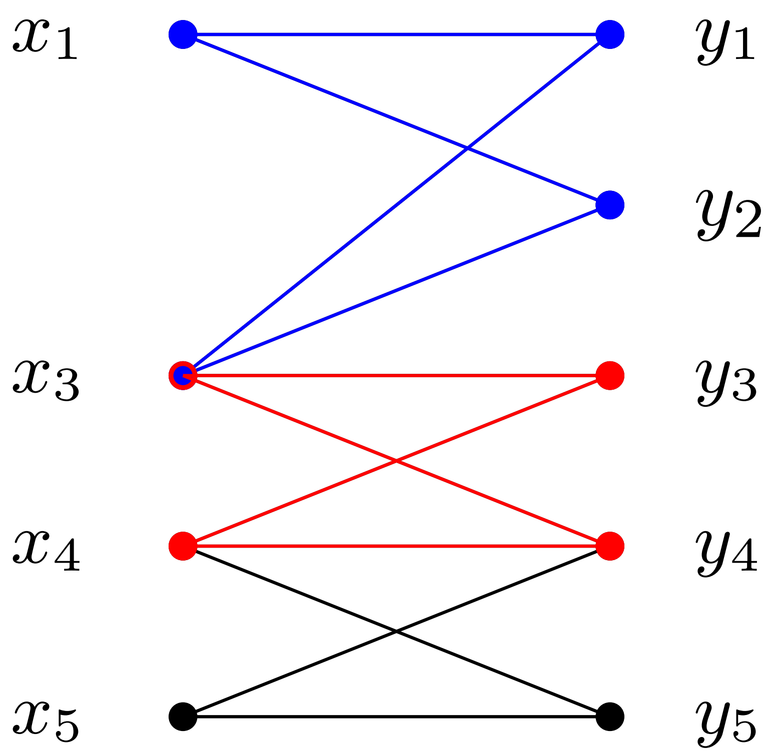
Let $\langle G, L, k \rangle$ be an instance of **EBC**.

Definition. An irrelevant vertex is incident to only edges in L.

Irrelevant Vertex Rule. Remove any irrelevant vertex from G.

Lemma. The Irrelevant Vertex Rule is safe.

$$\langle G, L, k \rangle \to \langle G \backslash \{x_2\}, L \cap E(G \backslash \{x_2\}), k \rangle$$



$(\{x_1, x_3\}, \{y_1, y_2\})$

$(\{x_3, x_4\}, \{y_3, y_4\})$

Irrelevant Vertex Rule

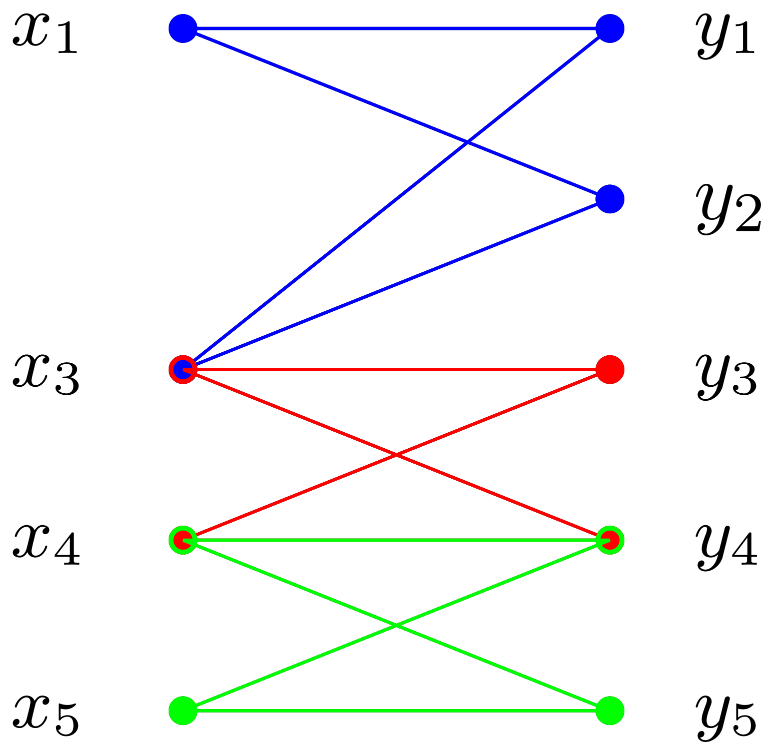
Let $\langle G, L, k \rangle$ be an instance of **EBC**.

Definition. An irrelevant vertex is incident to only edges in L.

Irrelevant Vertex Rule. Remove any irrelevant vertex from G.

Lemma. The Irrelevant Vertex Rule is safe.

$$\langle G, L, k \rangle \rightarrow \langle G \backslash \{x_2\}, L \cap E(G \backslash \{x_2\}), k \rangle$$



$(\{x_1, x_3\}, \{y_1, y_2\})$

$(\{x_3, x_4\}, \{y_3, y_4\})$

$(\{x_4, x_5\}, \{y_4, y_5\})$

Irrelevant Vertex Rule

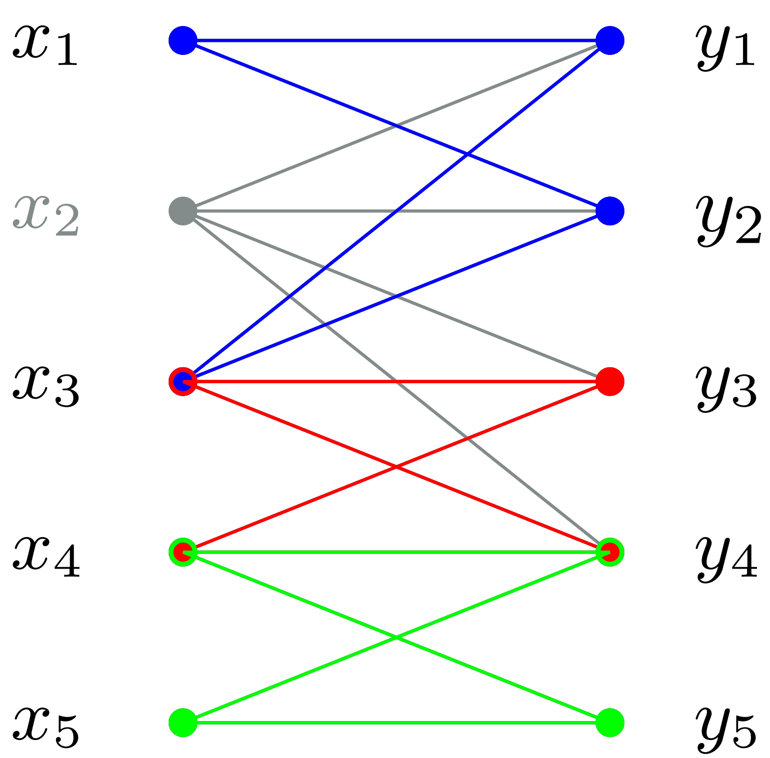
Let $\langle G, L, k\rangle$ be an instance of **EBC**.

Definition. An irrelevant vertex is incident to only edges in L.

Irrelevant Vertex Rule. Remove any irrelevant vertex from G.

Lemma. The Irrelevant Vertex Rule is safe.

$$\langle G, L, k\rangle \rightarrow \langle G\backslash\{x_2\}, L \cap E(G\backslash\{x_2\}), k\rangle$$



$(\{x_1, x_3\}, \{y_1, y_2\})$

$(\{x_3, x_4\}, \{y_3, y_4\})$

$(\{x_4, x_5\}, \{y_4, y_5\})$

Prisoner Rule

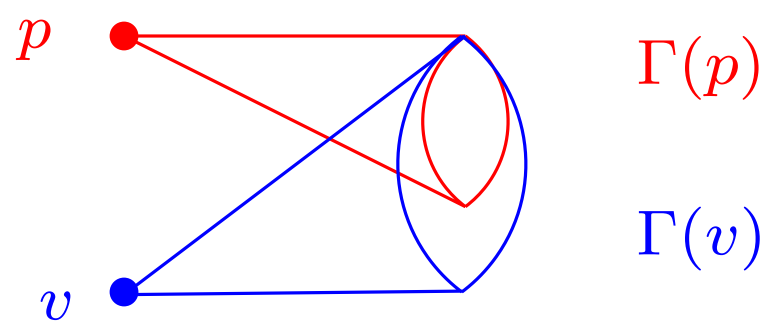
Let $\langle G, L, k \rangle$ be an instance of **EBC**.

Definition. A vertex p is a prisoner of a vertex v if $\Gamma(p) \subseteq \Gamma(v)$.

Prisoner Rule

Let $\langle G, L, k \rangle$ be an instance of **EBC**.

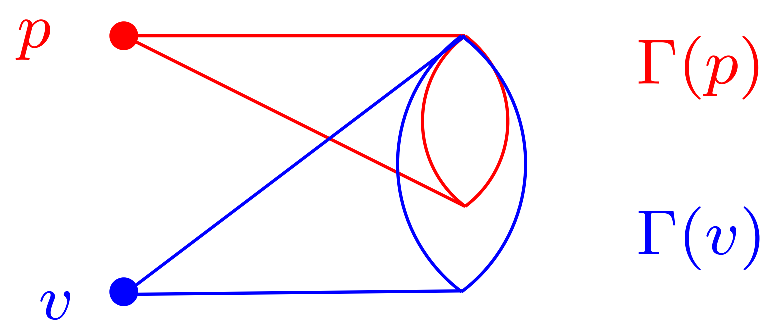
Definition. A vertex p is a prisoner of a vertex v if $\Gamma(p) \subseteq \Gamma(v)$.



Prisoner Rule

Let $\langle G, L, k \rangle$ be an instance of **EBC**.

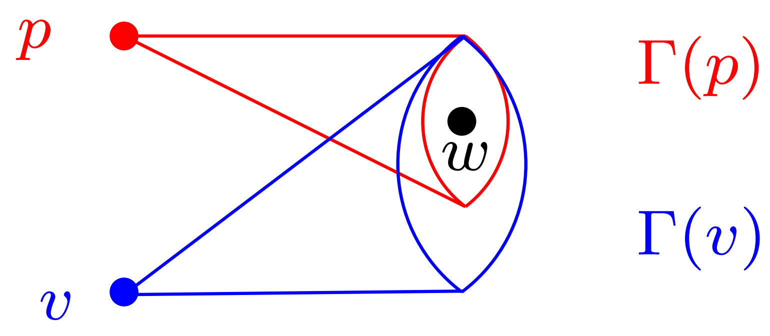
Definition. A vertex p is a prisoner of a vertex v if $\Gamma(p) \subseteq \Gamma(v)$.



Prisoner Rule. If (p, w) is not covered, mark (v, w) as covered.

Prisoner Rule

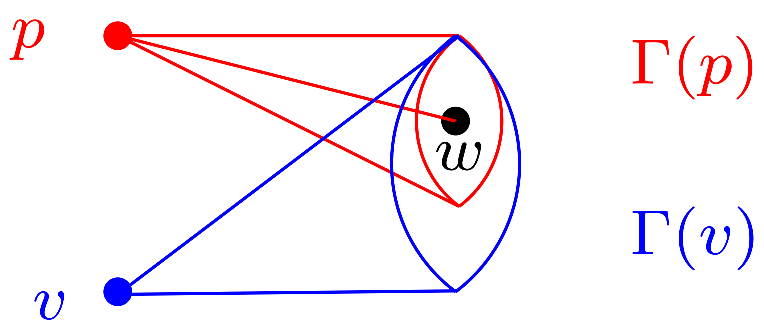
Let $\langle G, L, k \rangle$ be an instance of **EBC**.
Definition. A vertex p is a prisoner of a vertex v if $\Gamma(p) \subseteq \Gamma(v)$.



Prisoner Rule. If (p, w) is not covered, mark (v, w) as covered.

Prisoner Rule

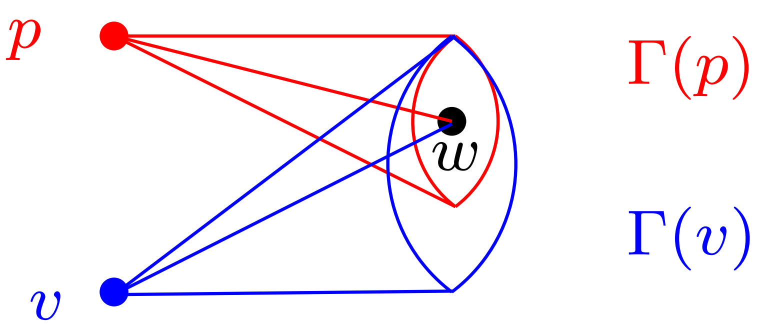
Let $\langle G, L, k \rangle$ be an instance of **EBC**.
Definition. A vertex p is a prisoner of a vertex v if $\Gamma(p) \subseteq \Gamma(v)$.



Prisoner Rule. If (p, w) is not covered, mark (v, w) as covered.

Prisoner Rule

Let $\langle G, L, k \rangle$ be an instance of **EBC**.
Definition. A vertex p is a prisoner of a vertex v if $\Gamma(p) \subseteq \Gamma(v)$.

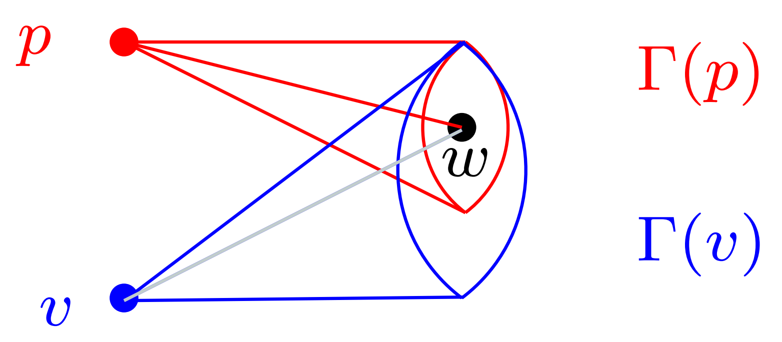


Prisoner Rule. If (p, w) is not covered, mark (v, w) as covered.

Prisoner Rule

Let $\langle G, L, k\rangle$ be an instance of **EBC**.

Definition. A vertex p is a prisoner of a vertex v if $\Gamma(p) \subseteq \Gamma(v)$.

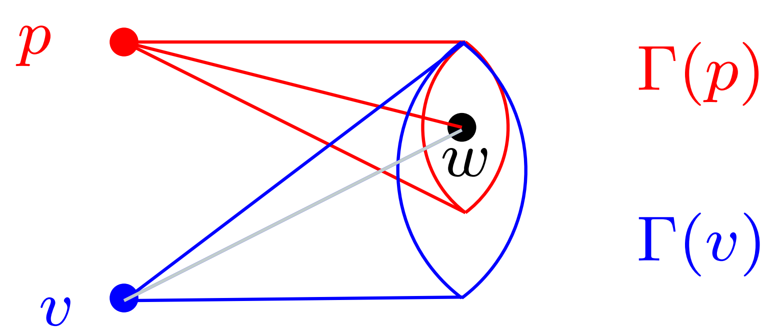


Prisoner Rule. If (p, w) is not covered, mark (v, w) as covered.

Prisoner Rule

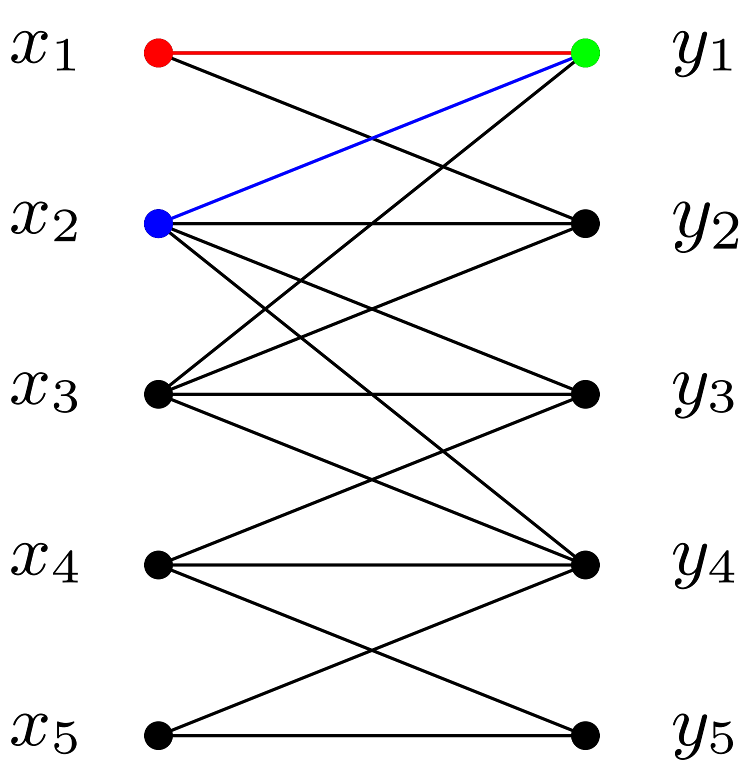
Let $\langle G, L, k \rangle$ be an instance of **EBC**.

Definition. A vertex p is a prisoner of a vertex v if $\Gamma(p) \subseteq \Gamma(v)$.



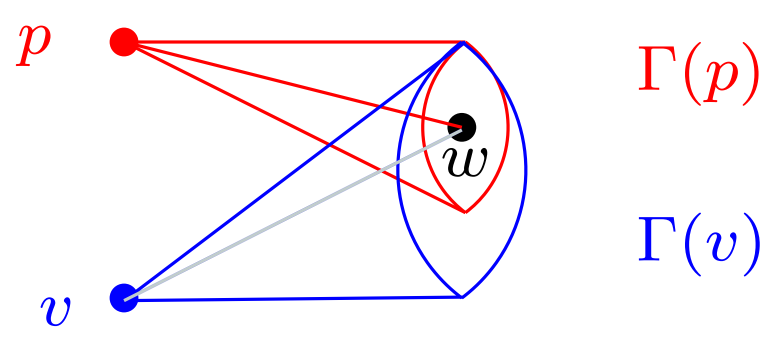
Prisoner Rule. If (p, w) is not covered, mark (v, w) as covered.

Lemma. The Prisoner Rule is safe.



Prisoner Rule

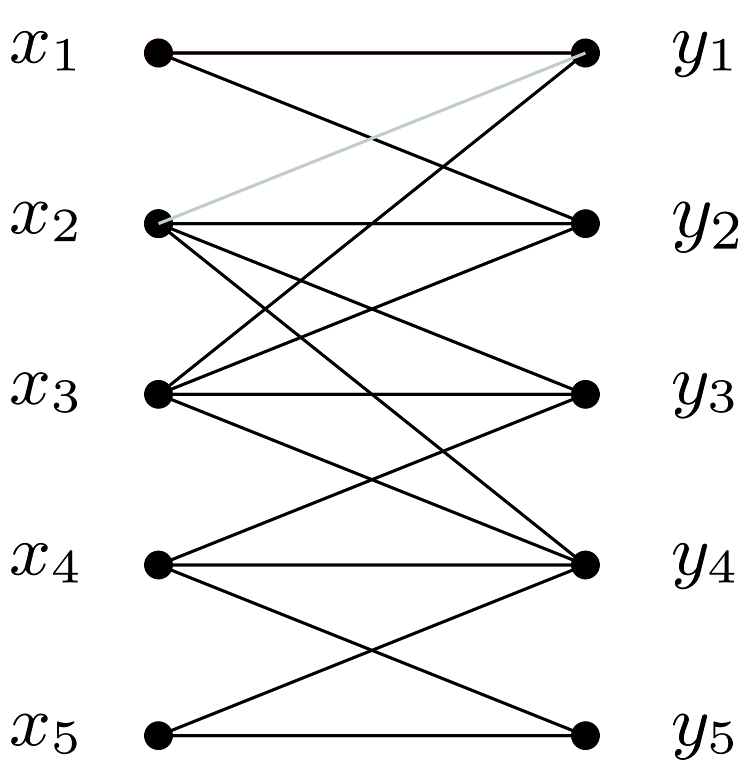
Let $\langle G, L, k\rangle$ be an instance of **EBC**.
Definition. A vertex p is a prisoner of a vertex v if $\Gamma(p) \subseteq \Gamma(v)$.



Prisoner Rule. If (p, w) is not covered, mark (v, w) as covered.

Lemma. The Prisoner Rule is safe.

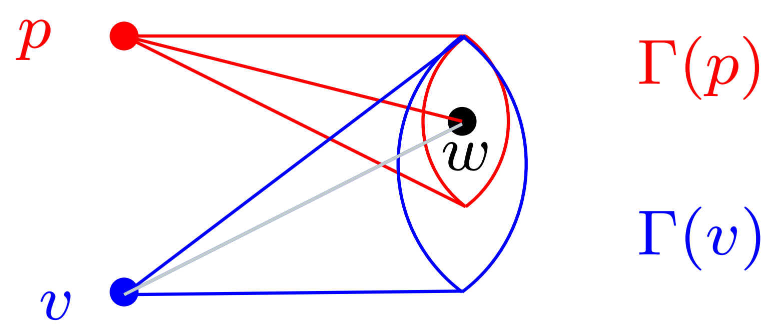
$$\langle G, L, k\rangle \rightarrow \langle G, L \cup \{(x_2, y_1)\}, k\rangle$$



Prisoner Rule

Let $\langle G, L, k\rangle$ be an instance of **EBC**.

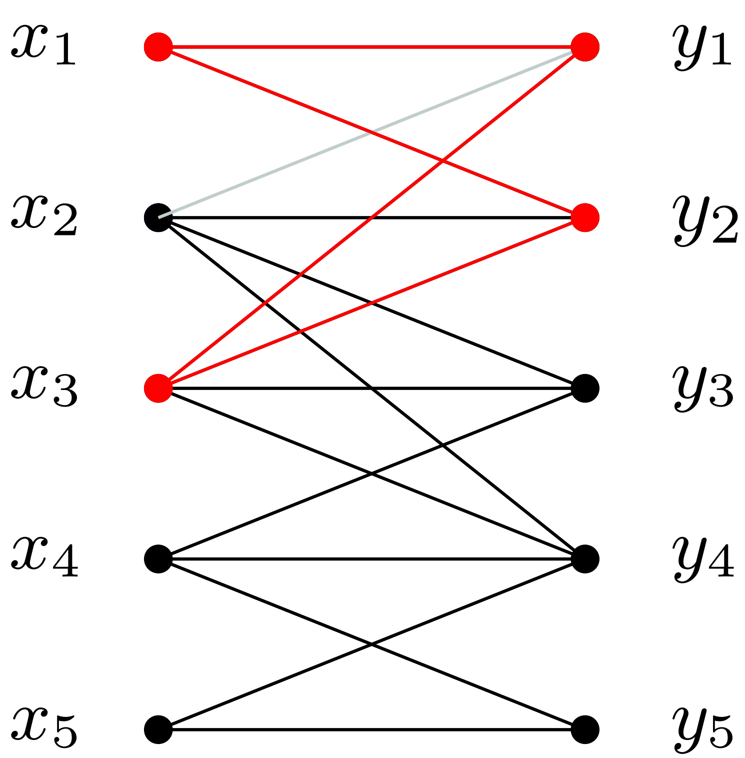
Definition. A vertex p is a prisoner of a vertex v if $\Gamma(p) \subseteq \Gamma(v)$.



Prisoner Rule. If (p, w) is not covered, mark (v, w) as covered.

Lemma. The Prisoner Rule is safe.

$$\langle G, L, k\rangle \rightarrow \langle G, L \cup \{(x_2, y_1)\}, k\rangle$$

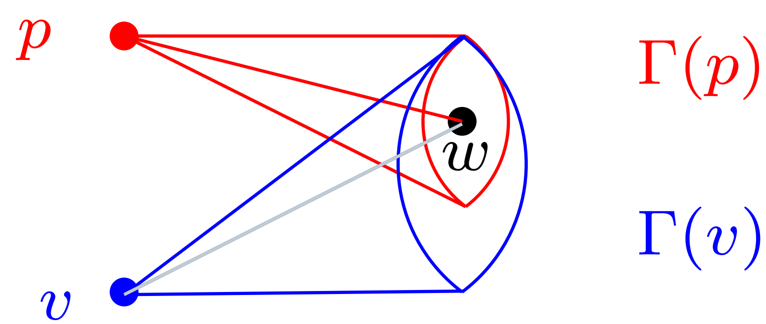


$(\{x_1, x_3\}, \{y_1, y_2\})$

Prisoner Rule

Let $\langle G, L, k \rangle$ be an instance of **EBC**.

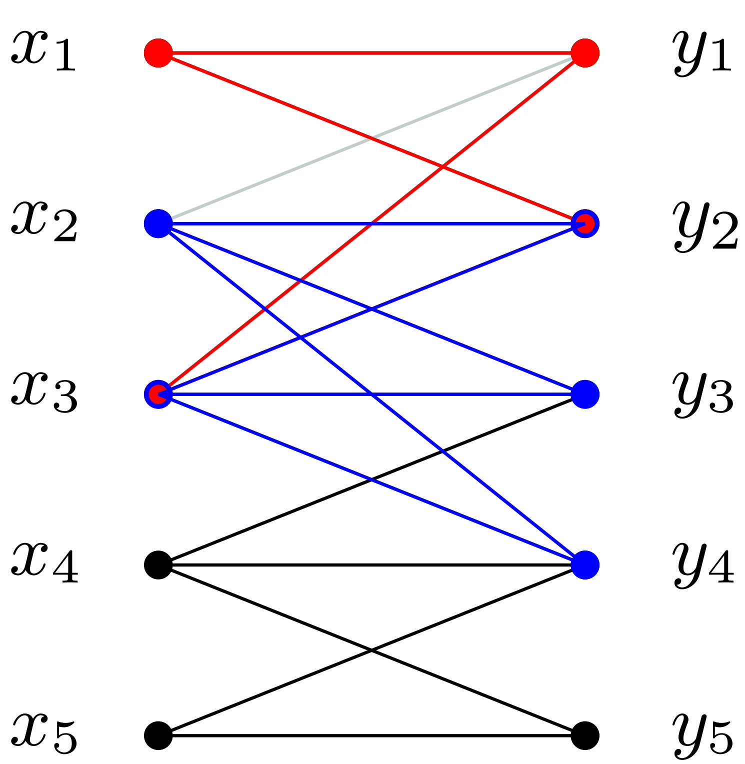
Definition. A vertex p is a prisoner of a vertex v if $\Gamma(p) \subseteq \Gamma(v)$.



Prisoner Rule. If (p, w) is not covered, mark (v, w) as covered.

Lemma. The Prisoner Rule is safe.

$$\langle G, L, k \rangle \to \langle G, L \cup \{(x_2, y_1)\}, k \rangle$$



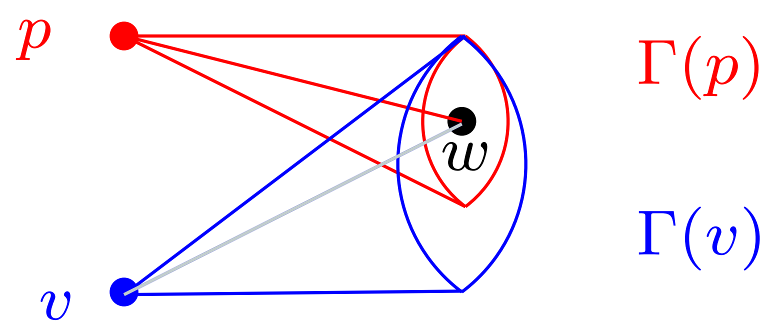
$(\{x_1, x_3\}, \{y_1, y_2\})$

$(\{x_2, x_3\}, \{y_2, y_3, y_4\})$

Prisoner Rule

Let $\langle G, L, k \rangle$ be an instance of **EBC**.

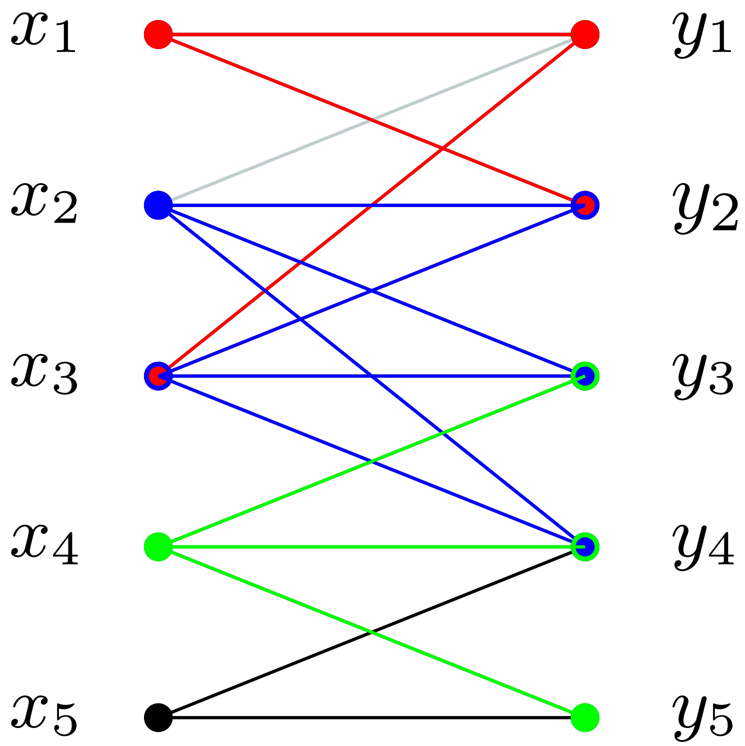
Definition. A vertex p is a prisoner of a vertex v if $\Gamma(p) \subseteq \Gamma(v)$.



Prisoner Rule. If (p, w) is not covered, mark (v, w) as covered.

Lemma. The Prisoner Rule is safe.

$$\langle G, L, k \rangle \rightarrow \langle G, L \cup \{(x_2, y_1)\}, k \rangle$$



$(\{x_1, x_3\}, \{y_1, y_2\})$

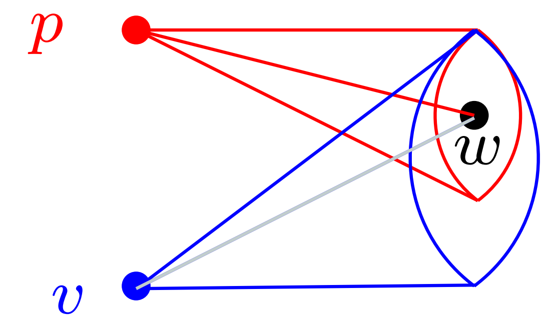
$(\{x_2, x_3\}, \{y_2, y_3, y_4\})$

$(\{x_4\}, \{y_3, y_4, y_5\})$

Prisoner Rule

Let $\langle G, L, k\rangle$ be an instance of **EBC**.

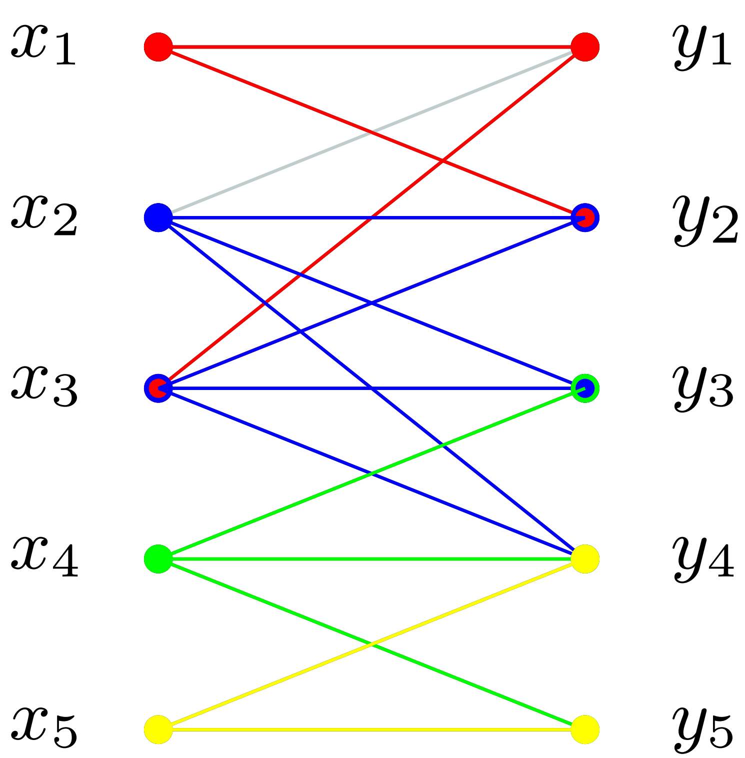
Definition. A vertex p is a prisoner of a vertex v if $\Gamma(p) \subseteq \Gamma(v)$.



Prisoner Rule. If (p, w) is not covered, mark (v, w) as covered.

Lemma. The Prisoner Rule is safe.

$$\langle G, L, k\rangle \rightarrow \langle G, L \cup \{(x_2, y_1)\}, k\rangle$$



$(\{x_1, x_3\}, \{y_1, y_2\})$

$(\{x_2, x_3\}, \{y_2, y_3, y_4\})$

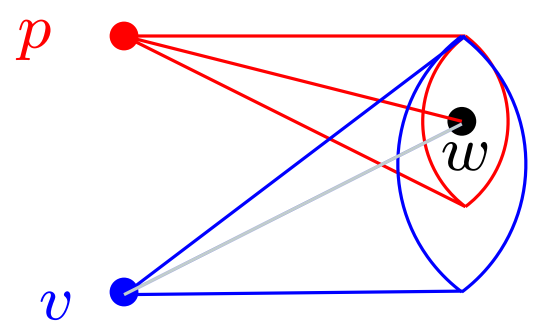
$(\{x_4\}, \{y_3, y_4, y_5\})$

$(\{x_5\}, \{y_4, y_5\})$

Prisoner Rule

Let $\langle G, L, k\rangle$ be an instance of **EBC**.

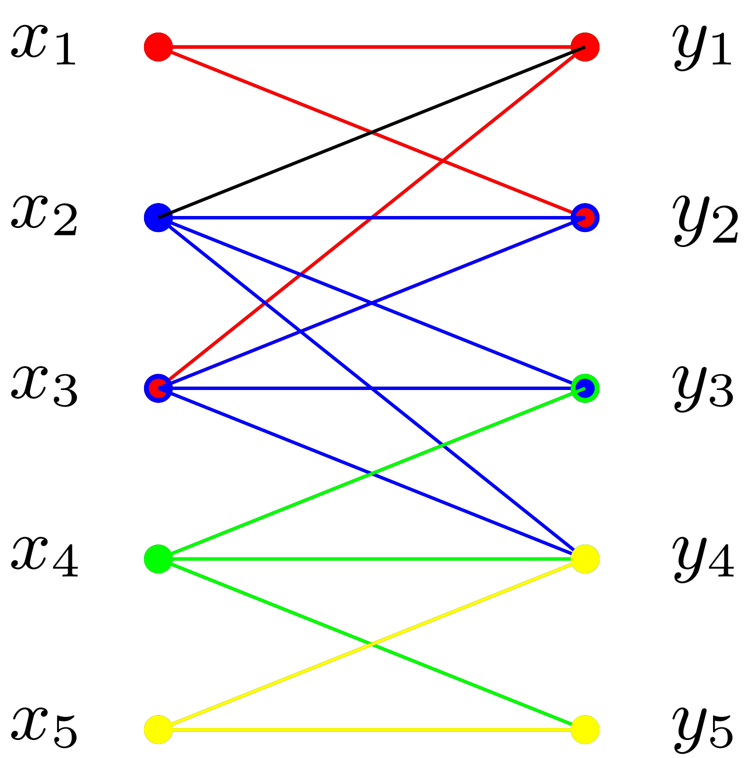
Definition. A vertex p is a prisoner of a vertex v if $\Gamma(p) \subseteq \Gamma(v)$.



Prisoner Rule. If (p, w) is not covered, mark (v, w) as covered.

Lemma. The Prisoner Rule is safe.

$$\langle G, L, k\rangle \rightarrow \langle G, L \cup \{(x_2, y_1)\}, k\rangle$$



$(\{x_1, x_3\}, \{y_1, y_2\})$

$(\{x_2, x_3\}, \{y_2, y_3, y_4\})$

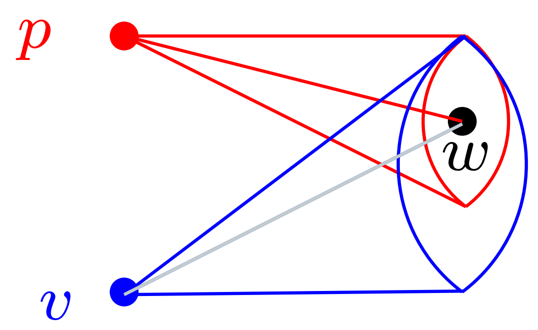
$(\{x_4\}, \{y_3, y_4, y_5\})$

$(\{x_5\}, \{y_4, y_5\})$

Prisoner Rule

Let $\langle G, L, k \rangle$ be an instance of **EBC**.

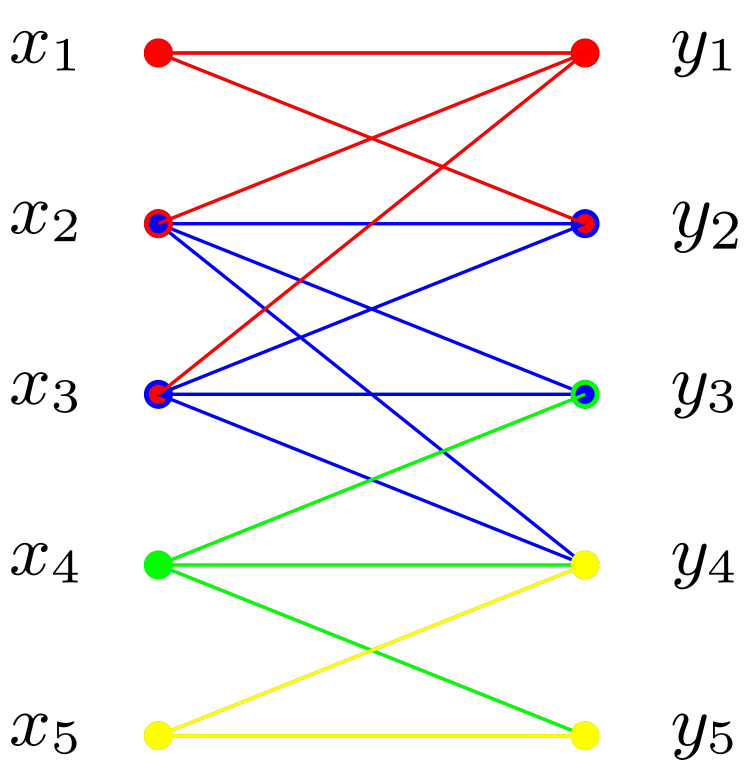
Definition. A vertex p is a prisoner of a vertex v if $\Gamma(p) \subseteq \Gamma(v)$.



Prisoner Rule. If (p, w) is not covered, mark (v, w) as covered.

Lemma. The Prisoner Rule is safe.

$$\langle G, L, k \rangle \rightarrow \langle G, L \cup \{(x_2, y_1)\}, k \rangle$$



$(\{x_1, x_2, x_3\}, \{y_1, y_2\})$

$(\{x_2, x_3\}, \{y_2, y_3, y_4\})$

$(\{x_4\}, \{y_3, y_4, y_5\})$

$(\{x_5\}, \{y_4, y_5\})$

Bisimplicial Edge Rule

Let $\langle G, L, k \rangle$ be an instance of **EBC**.

Definition. An edge (u, v) is bisimplicial if and only if $(\Gamma(v), \Gamma(u))$ induces a biclique in G.

Bisimplicial Edge Rule

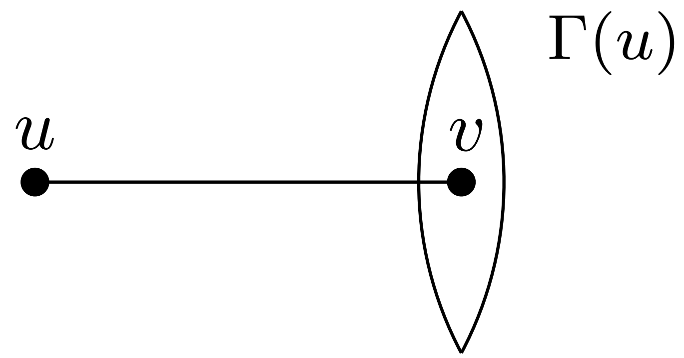
Let $\langle G, L, k \rangle$ be an instance of **EBC**.

Definition. An edge (u, v) is bisimplicial if and only if $(\Gamma(v), \Gamma(u))$ induces a biclique in G.

Bisimplicial Edge Rule

Let $\langle G, L, k \rangle$ be an instance of **EBC**.

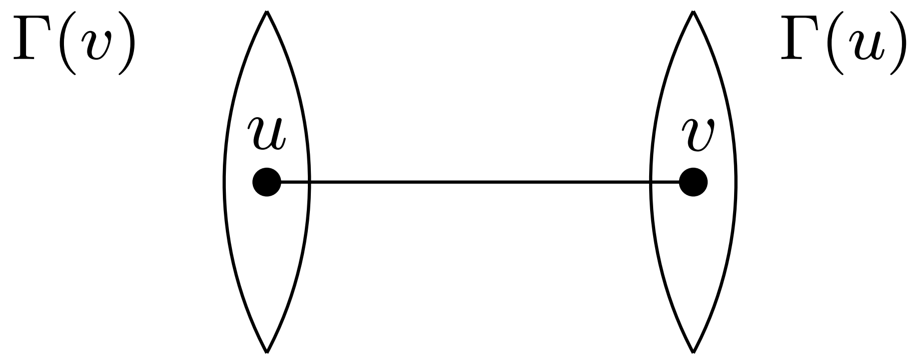
Definition. An edge (u, v) is bisimplicial if and only if $(\Gamma(v), \Gamma(u))$ induces a biclique in G.



Bisimplicial Edge Rule

Let $\langle G, L, k \rangle$ be an instance of **EBC**.

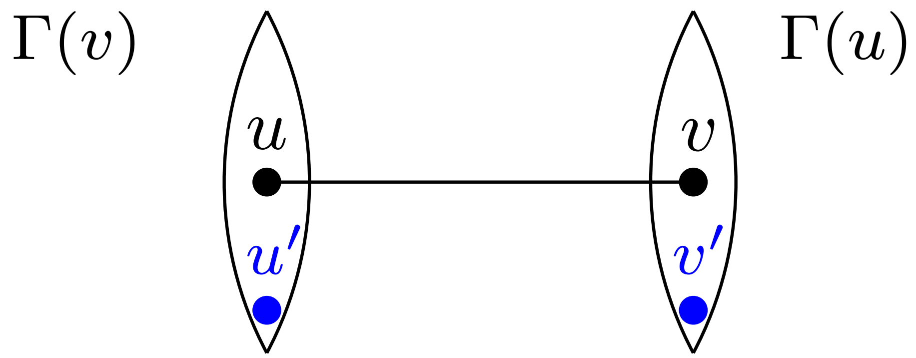
Definition. An edge (u, v) is bisimplicial if and only if $(\Gamma(v), \Gamma(u))$ induces a biclique in G.



Bisimplicial Edge Rule

Let $\langle G, L, k \rangle$ be an instance of **EBC**.

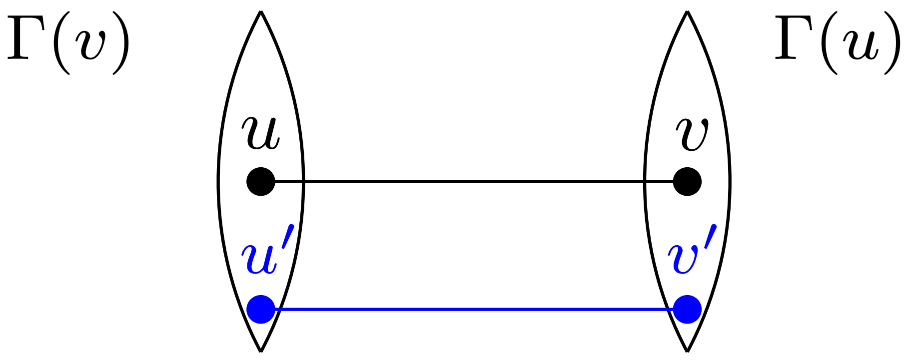
Definition. An edge (u, v) is bisimplicial if and only if $(\Gamma(v), \Gamma(u))$ induces a biclique in G.



Bisimplicial Edge Rule

Let $\langle G, L, k \rangle$ be an instance of **EBC**.

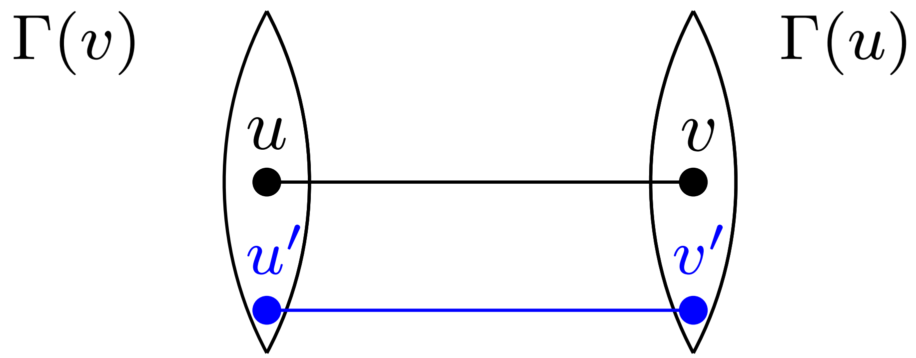
Definition. An edge (u, v) is bisimplicial if and only if $(\Gamma(v), \Gamma(u))$ induces a biclique in G.



Bisimplicial Edge Rule

Let $\langle G, L, k \rangle$ be an instance of **EBC**.

Definition. An edge (u, v) is bisimplicial if and only if $(\Gamma(v), \Gamma(u))$ induces a biclique in G.

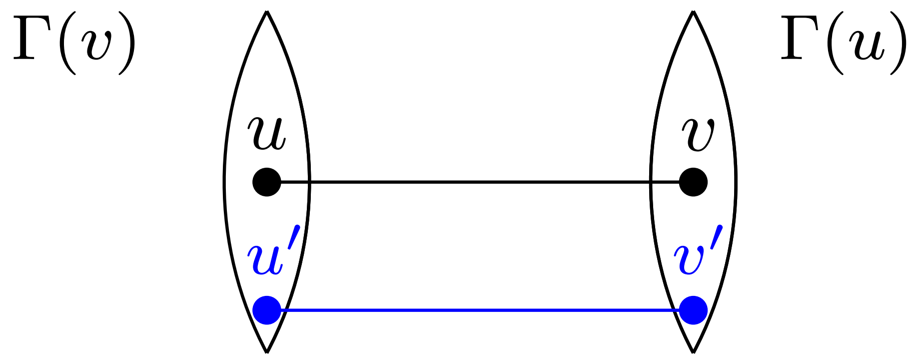


Bisimplicial Edge Rule. If an uncovered edge (u, v) is bisimplicial, add the biclique $B = (\Gamma(v), \Gamma(u))$ to the solution and mark its edges as covered.

Bisimplicial Edge Rule

Let $\langle G, L, k \rangle$ be an instance of **EBC**.

Definition. An edge (u, v) is bisimplicial if and only if $(\Gamma(v), \Gamma(u))$ induces a biclique in G.



Bisimplicial Edge Rule. If an uncovered edge (u, v) is bisimplicial, add the biclique $B = (\Gamma(v), \Gamma(u))$ to the solution and mark its edges as covered.

Lemma. If (u, v) is bisimplicial, the biclique $B = (\Gamma(v), \Gamma(u))$ is the unique maximal biclique that covers the edge (u, v).

Corollary. The Bisimplicial Edge Rule is safe.

Bisimplicial Edge Rule

Let $\langle G, L, k \rangle$ be an instance of **EBC**.

Bisimplicial Edge Rule. If an uncovered edge (u, v) is bisimplicial, add the biclique $B = (\Gamma(v), \Gamma(u))$ to the solution and mark its edges as covered.

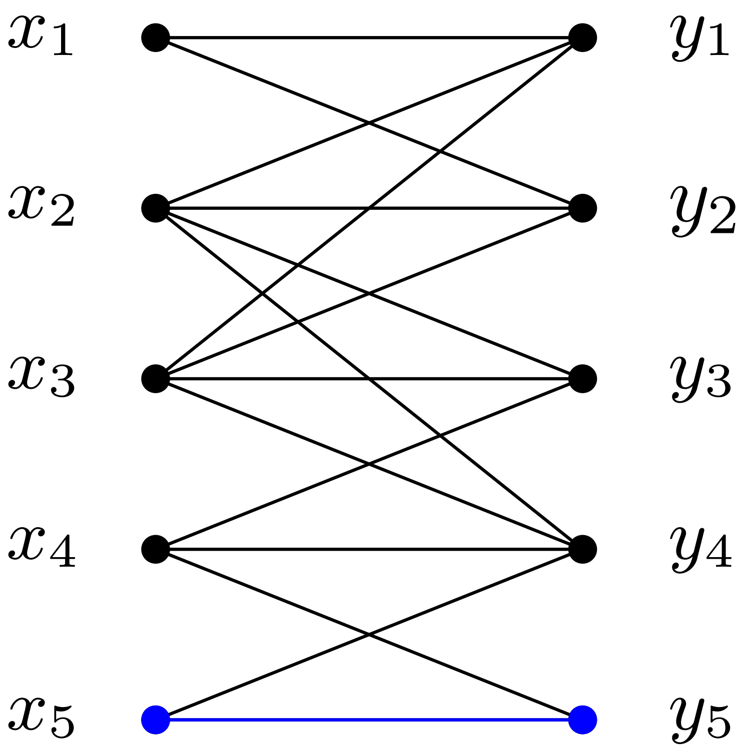
Lemma. The Bisimplicial Edge Rule is safe.

Bisimplicial Edge Rule

Let $\langle G, L, k\rangle$ be an instance of **EBC**.

Bisimplicial Edge Rule. If an uncovered edge (u, v) is bisimplicial, add the biclique $B = (\Gamma(v), \Gamma(u))$ to the solution and mark its edges as covered.

Lemma. The Bisimplicial Edge Rule is safe.



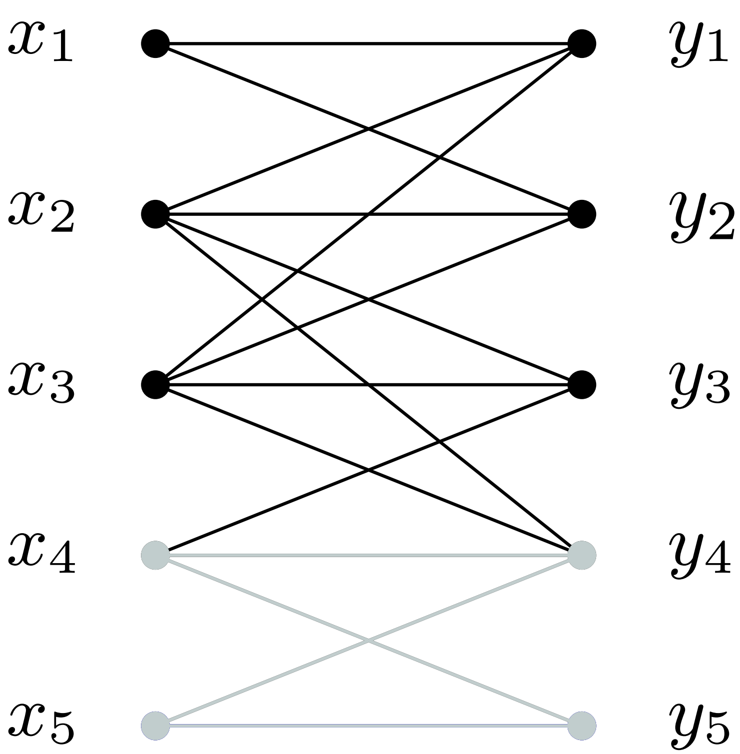
Bisimplicial Edge Rule

Let $\langle G, L, k\rangle$ be an instance of **EBC**.

Bisimplicial Edge Rule. If an uncovered edge (u, v) is bisimplicial, add the biclique $B = (\Gamma(v), \Gamma(u))$ to the solution and mark its edges as covered.

Lemma. The Bisimplicial Edge Rule is safe.

$$\langle G, L, k\rangle \rightarrow \langle G, L \cup E(B), k-1\rangle$$



$(\{x_4, x_5\}, \{y_4, y_5\})$

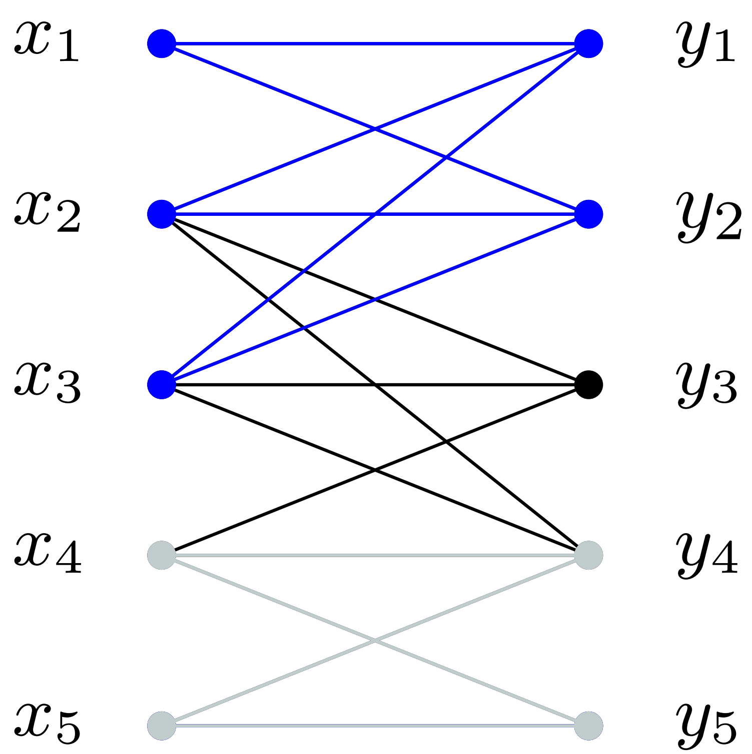
Bisimplicial Edge Rule

Let $\langle G, L, k \rangle$ be an instance of **EBC**.

Bisimplicial Edge Rule. If an uncovered edge (u, v) is bisimplicial, add the biclique $B = (\Gamma(v), \Gamma(u))$ to the solution and mark its edges as covered.

Lemma. The Bisimplicial Edge Rule is safe.

$$\langle G, L, k \rangle \rightarrow \langle G, L \cup E(B), k - 1 \rangle$$



$(\{x_4, x_5\}, \{y_4, y_5\})$

$(\{x_1, x_2, x_3\}, \{y_1, y_2\})$

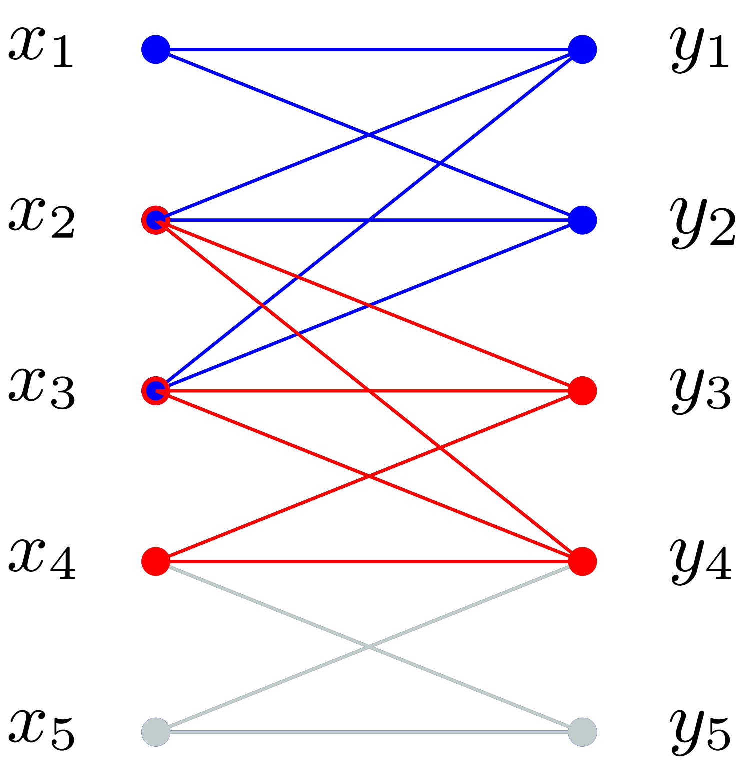
Bisimplicial Edge Rule

Let $\langle G, L, k \rangle$ be an instance of **EBC**.

Bisimplicial Edge Rule. If an uncovered edge (u, v) is bisimplicial, add the biclique $B = (\Gamma(v), \Gamma(u))$ to the solution and mark its edges as covered.

Lemma. The Bisimplicial Edge Rule is safe.

$$\langle G, L, k \rangle \rightarrow \langle G, L \cup E(B), k-1 \rangle$$



$(\{x_4, x_5\}, \{y_4, y_5\})$

$(\{x_1, x_2, x_3\}, \{y_1, y_2\})$

$(\{x_2, x_3, x_4\}, \{y_3, y_4\})$

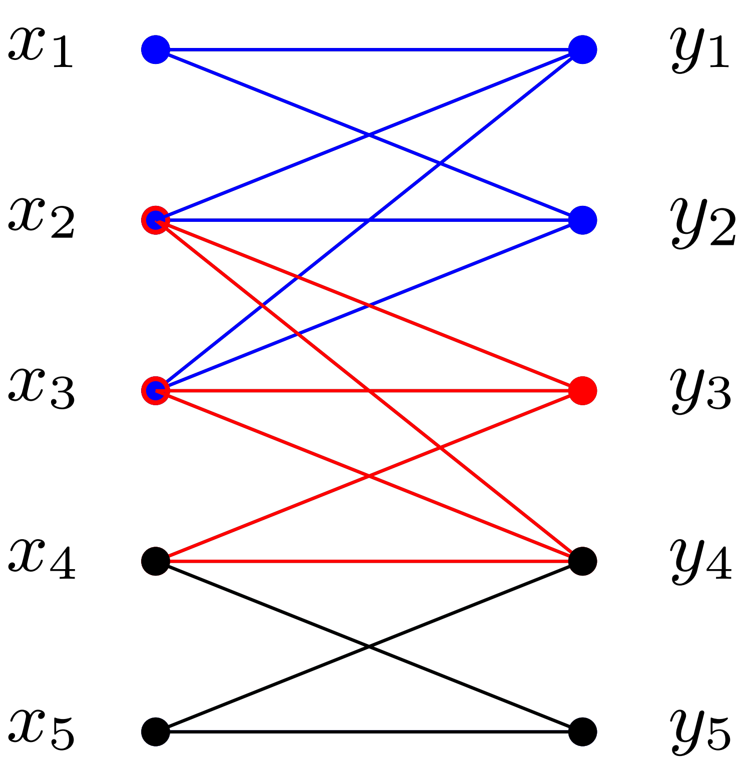
Bisimplicial Edge Rule

Let $\langle G, L, k\rangle$ be an instance of **EBC**.

Bisimplicial Edge Rule. If an uncovered edge (u, v) is bisimplicial, add the biclique $B = (\Gamma(v), \Gamma(u))$ to the solution and mark its edges as covered.

Lemma. The Bisimplicial Edge Rule is safe.

$$\langle G, L, k\rangle \to \langle G, L \cup E(B), k-1\rangle$$



$$(\{x_4, x_5\}, \{y_4, y_5\})$$

$$(\{x_1, x_2, x_3\}, \{y_1, y_2\})$$

$$(\{x_2, x_3, x_4\}, \{y_3, y_4\})$$

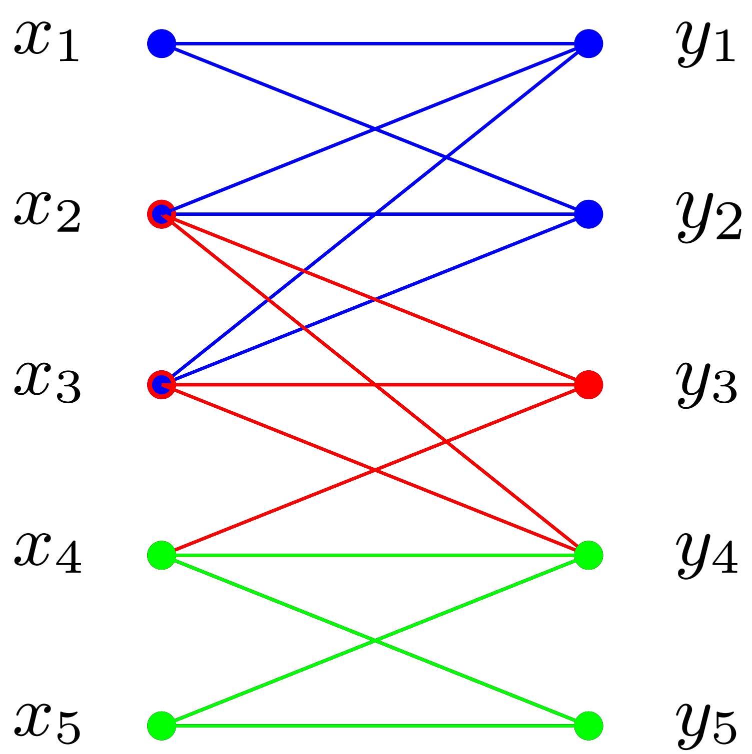
Bisimplicial Edge Rule

Let $\langle G, L, k \rangle$ be an instance of **EBC**.

Bisimplicial Edge Rule. If an uncovered edge (u, v) is bisimplicial, add the biclique $B = (\Gamma(v), \Gamma(u))$ to the solution and mark its edges as covered.

Lemma. The Bisimplicial Edge Rule is safe.

$$\langle G, L, k \rangle \rightarrow \langle G, L \cup E(B), k - 1 \rangle$$



$(\{x_4, x_5\}, \{y_4, y_5\})$

$(\{x_1, x_2, x_3\}, \{y_1, y_2\})$

$(\{x_2, x_3, x_4\}, \{y_3, y_4\})$

Extended Bisimplicial Edge Rule

Let $\langle G, L, k\rangle$ be an instance of **EBC**.

Definition. Let $\mathcal{N}(u, v)$ denote the set of all edges $(x, y) \in E(G)\backslash L$ such that $x \in \Gamma(v)$ and $y \in \Gamma(u)$.

Definition. An edge (u, v) is an extended bisimplicial edge if and only if $\mathcal{N}(u, v)$ induces a biclique in G.

Extended Bisimplicial Edge Rule

Let $\langle G, L, k \rangle$ be an instance of **EBC**.

Definition. Let $\mathcal{N}(u,v)$ denote the set of all edges $(x,y) \in E(G) \backslash L$ such that $x \in \Gamma(v)$ and $y \in \Gamma(u)$.

Definition. An edge (u,v) is an extended bisimplicial edge if and only if $\mathcal{N}(u,v)$ induces a biclique in G.

Extended Bisimplicial Edge Rule. If an uncovered edge (u,v) is an extended bisimplicial edge, add the biclique $B = G[\mathcal{N}(u,v)]$ to the solution and mark its edges as covered.

Lemma. The Extended Bisimplicial Edge Rule is safe.

Extended Bisimplicial Edge Rule

Let $\langle G, L, k \rangle$ be an instance of **EBC**.

Extended Bisimplicial Edge Rule. If an uncovered edge (u, v) is an extended bisimplicial edge, add the biclique $B = G[\mathcal{N}(u, v)]$ to the solution and mark its edges as covered.

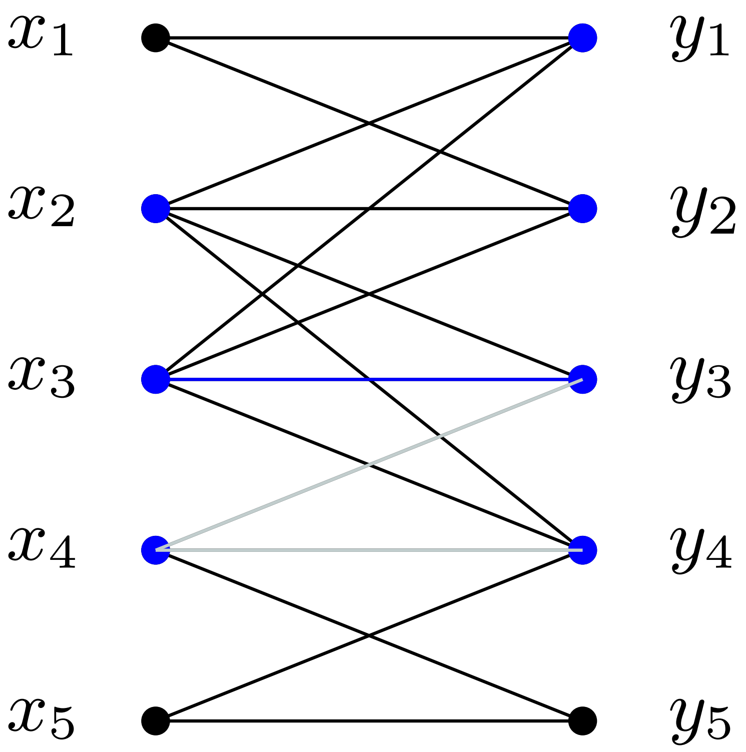
Lemma. The Extended Bisimplicial Edge Rule is safe.

Extended Bisimplicial Edge Rule

Let $\langle G, L, k \rangle$ be an instance of **EBC**.

Extended Bisimplicial Edge Rule. If an uncovered edge (u, v) is an extended bisimplicial edge, add the biclique $B = G[\mathcal{N}(u, v)]$ to the solution and mark its edges as covered.

Lemma. The Extended Bisimplicial Edge Rule is safe.



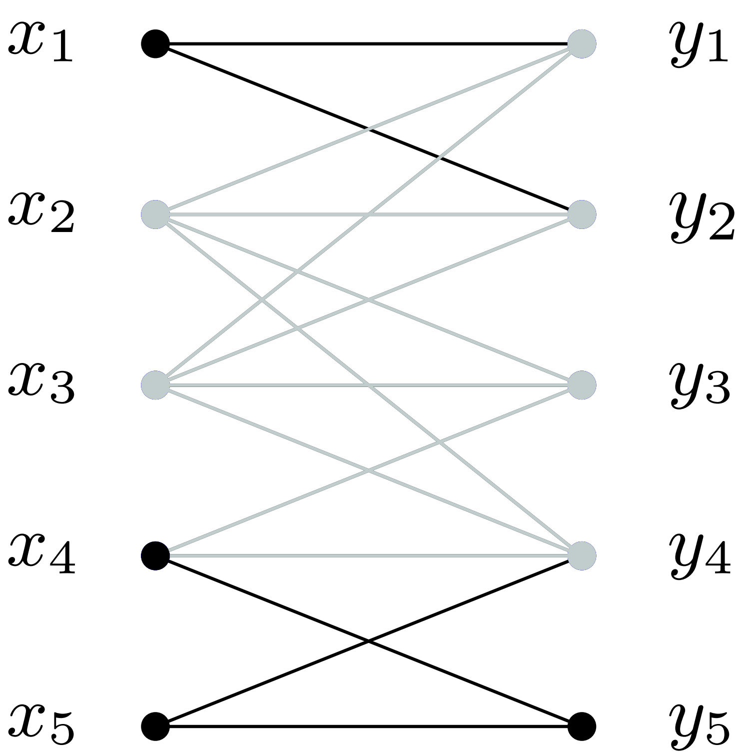
Extended Bisimplicial Edge Rule

Let $\langle G, L, k \rangle$ be an instance of **EBC**.

Extended Bisimplicial Edge Rule. If an uncovered edge (u, v) is an extended bisimplicial edge, add the biclique $B = G[\mathcal{N}(u, v)]$ to the solution and mark its edges as covered.

Lemma. The Extended Bisimplicial Edge Rule is safe.

$$\langle G, L, k \rangle \to \langle G, L \cup \mathcal{N}(x_3, y_3), k - 1 \rangle$$



$(\{x_2, x_3\}, \{y_1, y_2, y_3\})$

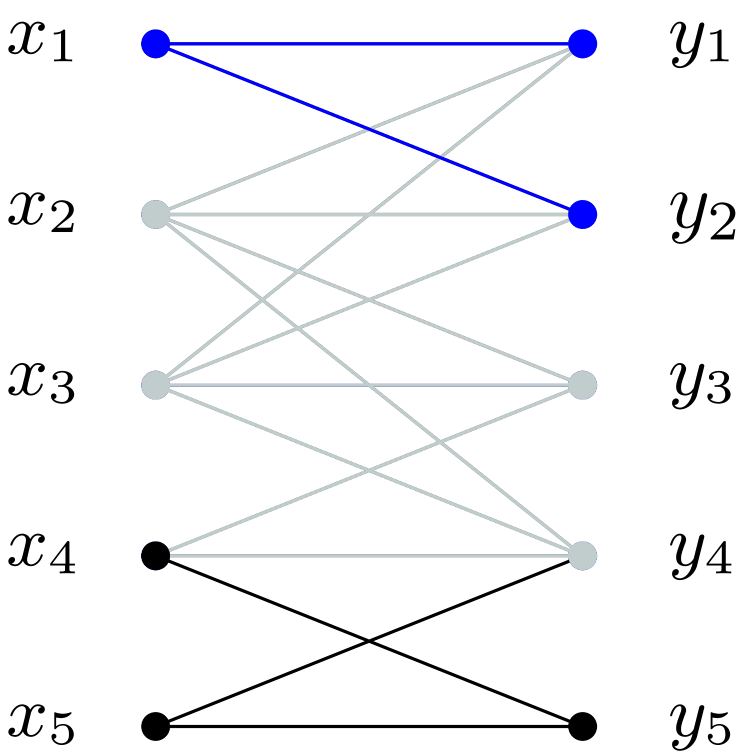
Extended Bisimplicial Edge Rule

Let $\langle G, L, k\rangle$ be an instance of **EBC**.

Extended Bisimplicial Edge Rule. If an uncovered edge (u, v) is an extended bisimplicial edge, add the biclique $B = G[\mathcal{N}(u, v)]$ to the solution and mark its edges as covered.

Lemma. The Extended Bisimplicial Edge Rule is safe.

$$\langle G, L, k\rangle \rightarrow \langle G, L \cup \mathcal{N}(x_3, y_3), k-1\rangle$$



$(\{x_2, x_3\}, \{y_1, y_2, y_3\})$

$(\{x_1\}, \{y_1, y_2\})$

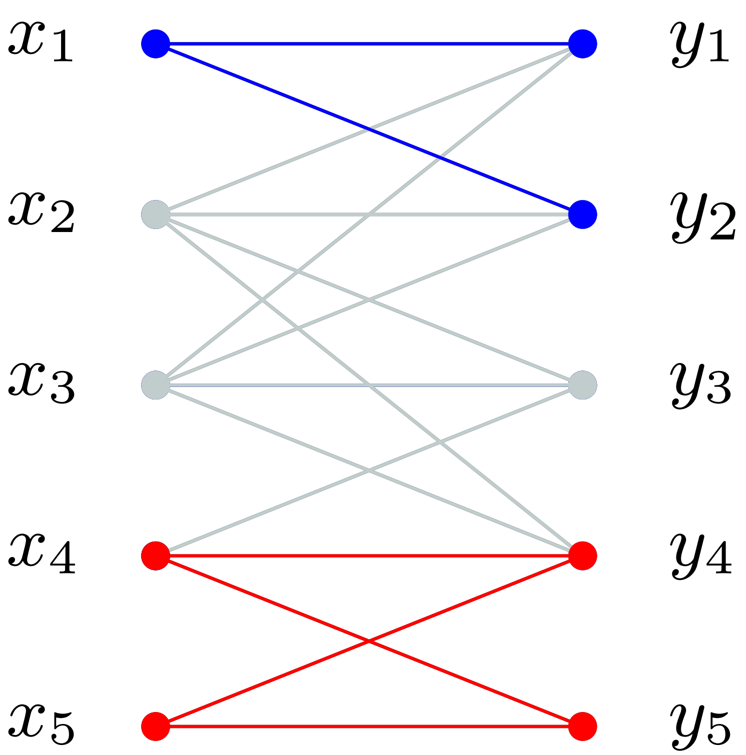
Extended Bisimplicial Edge Rule

Let $\langle G, L, k \rangle$ be an instance of **EBC**.

Extended Bisimplicial Edge Rule. If an uncovered edge (u, v) is an extended bisimplicial edge, add the biclique $B = G[\mathcal{N}(u, v)]$ to the solution and mark its edges as covered.

Lemma. The Extended Bisimplicial Edge Rule is safe.

$$\langle G, L, k \rangle \rightarrow \langle G, L \cup \mathcal{N}(x_3, y_3), k - 1 \rangle$$



$(\{x_2, x_3\}, \{y_1, y_2, y_3\})$

$(\{x_1\}, \{y_1, y_2\})$

$(\{x_4, x_5\}, \{y_4, y_5\})$

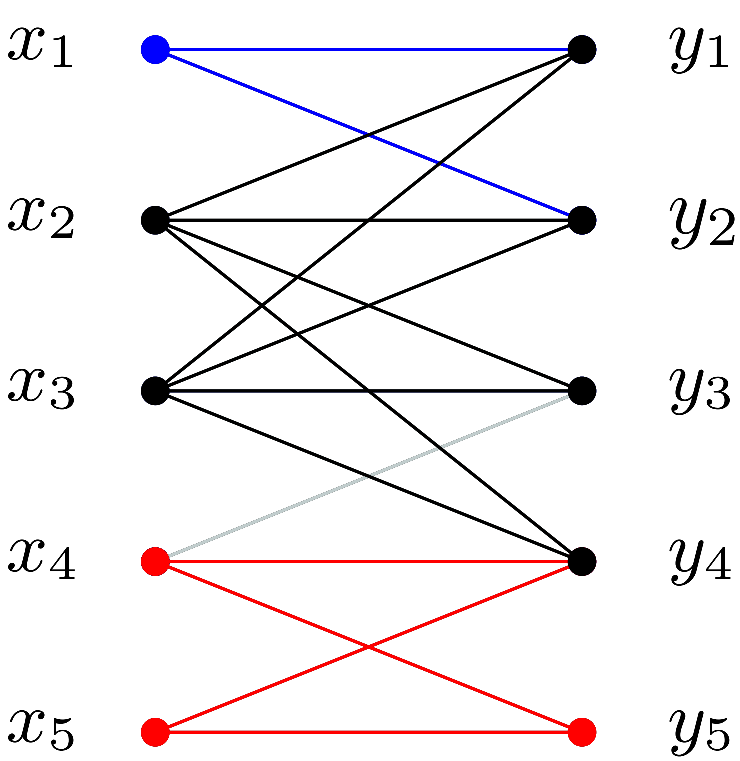
Extended Bisimplicial Edge Rule

Let $\langle G, L, k \rangle$ be an instance of **EBC**.

Extended Bisimplicial Edge Rule. If an uncovered edge (u, v) is an extended bisimplicial edge, add the biclique $B = G[\mathcal{N}(u, v)]$ to the solution and mark its edges as covered.

Lemma. The Extended Bisimplicial Edge Rule is safe.

$$\langle G, L, k \rangle \rightarrow \langle G, L \cup \mathcal{N}(x_3, y_3), k - 1 \rangle$$



$(\{x_2, x_3\}, \{y_1, y_2, y_3\})$

$(\{x_1\}, \{y_1, y_2\})$

$(\{x_4, x_5\}, \{y_4, y_5\})$

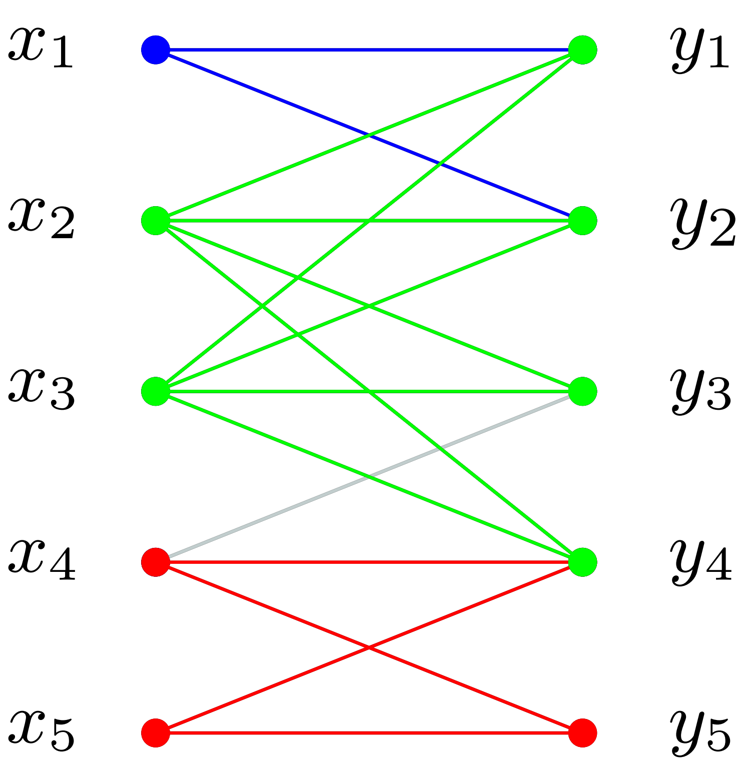
Extended Bisimplicial Edge Rule

Let $\langle G, L, k \rangle$ be an instance of **EBC**.

Extended Bisimplicial Edge Rule. If an uncovered edge (u, v) is an extended bisimplicial edge, add the biclique $B = G[\mathcal{N}(u, v)]$ to the solution and mark its edges as covered.

Lemma. The Extended Bisimplicial Edge Rule is safe.

$$\langle G, L, k \rangle \rightarrow \langle G, L \cup \mathcal{N}(x_3, y_3), k - 1 \rangle$$



$(\{x_2, x_3\}, \{y_1, y_2, y_3\})$

$(\{x_1\}, \{y_1, y_2\})$

$(\{x_4, x_5\}, \{y_4, y_5\})$

Reduction Results

Dataset	L	R	M	LB
domino	79	231	730	20
healthcare	46	46	1486	14
emea	35	3046	7220	34
apj	2044	1164	6841	453
usa	10962	285	45427	276
americas_small	3477	1587	105205	173
americas_large	3485	10127	185294	390

Reduction Results

Dataset	L	R	M	LB
domino	79	231	730	20
healthcare	46	46	1486	14
emea	35	3046	7220	34
apj	2044	1164	6841	453
usa	10962	285	45427	276
americas_small	3477	1587	105205	173
americas_large	3485	10127	185294	390

Running Time: less than 8 minutes.

Dataset	L'	R'	M'	M''	RC
domino	0	0	0	0	20
healthcare	0	0	0	0	14
emea	0	0	0	0	34
apj	0	0	0	0	453
usa	0	0	0	0	276
americas_small	29	24	233	44	159
americas_large	27	41	209	97	371

Reduction Results

Dataset	L	R	M	LB
domino	79	231	730	20
healthcare	46	46	1486	14
emea	35	3046	7220	34
apj	2044	1164	6841	453
usa	10962	285	45427	276
americas_small	3477	1587	105205	173
americas_large	3485	10127	185294	390

Running Time: less than 8 minutes.

Dataset	L'	R'	M'	M''	RC
domino	0	0	0	0	20
healthcare	0	0	0	0	14
emea	0	0	0	0	34
apj	0	0	0	0	453
usa	0	0	0	0	276
americas_small	29	24	233	44	159
americas_large	27	41	209	97	371

Reduction Results

Dataset	L	R	M	LB
domino	79	231	730	20
healthcare	46	46	1486	14
emea	35	3046	7220	34
apj	2044	1164	6841	453
usa	10962	285	45427	276
americas_small	3477	1587	105205	173
americas_large	3485	10127	185294	390

Running Time: less than 8 minutes.

Dataset	L'	R'	M'	M''	RC
domino	0	0	0	0	20
healthcare	0	0	0	0	14
emea	0	0	0	0	34
apj	0	0	0	0	453
usa	0	0	0	0	276
americas_small	29	24	233	44	159
americas_large	27	41	209	97	371

Chromatic Characterization

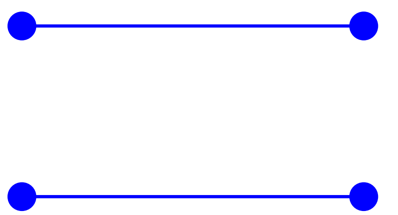
Let G be a bipartite graph.

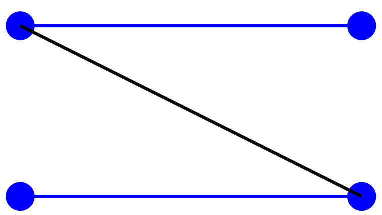
Definition. Two edges are independent if they are vertex disjoint and they do not belong to a common C_4.

Chromatic Characterization

Let G be a bipartite graph.

Definition. Two edges are independent if they are vertex disjoint and they do not belong to a common C_4.

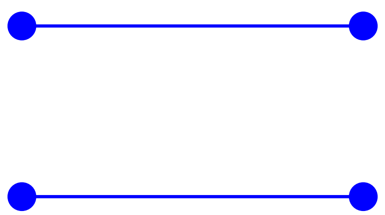

$\overline{C_4}$


P_3

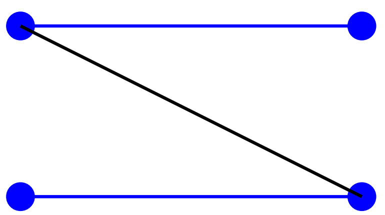
Chromatic Characterization

Let G be a bipartite graph.

Definition. Two edges are independent if they are vertex disjoint and they do not belong to a common C_4.



$\overline{C_4}$



P_3

Edge Colouring. No two monochromatic edges are independent.

Lemma. The edges of G can be covered with k bicliques if and only if the edges of G can be coloured with k colours.

Chromatic Characterization

Let G be a bipartite graph.

Edge Colouring. No two monochromatic edges are independent.

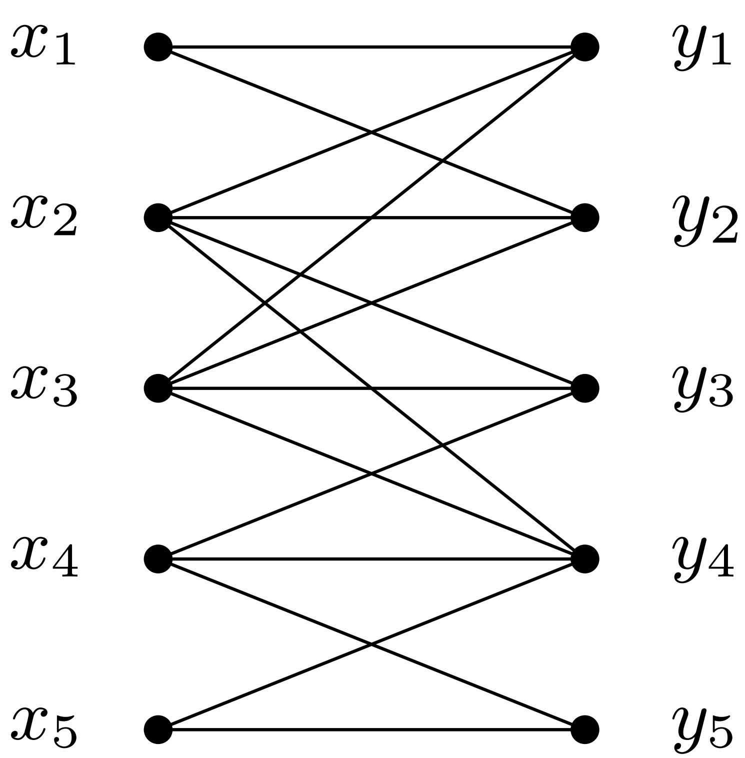
Lemma. The edges of G can be covered with k bicliques if and only if the edges of G can be coloured with k colours.

Chromatic Characterization

Let G be a bipartite graph.

Edge Colouring. No two monochromatic edges are independent.

Lemma. The edges of G can be covered with k bicliques if and only if the edges of G can be coloured with k colours.



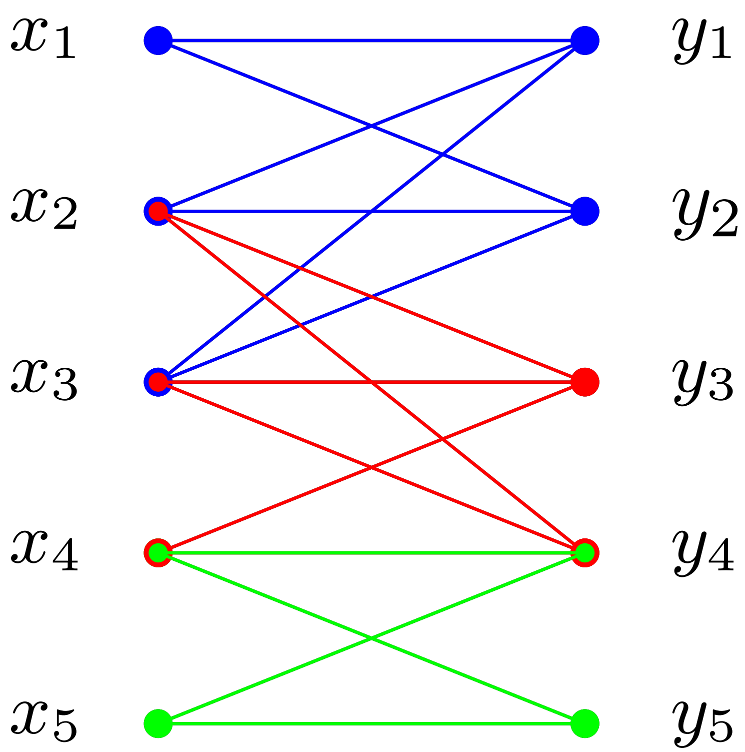
Chromatic Characterization

Let G be a bipartite graph.

Edge Colouring. No two monochromatic edges are independent.

Lemma. The edges of G can be covered with k bicliques if and only if the edges of G can be coloured with k colours.

$$\{(\{x_1, x_2, x_3\}, \{y_1, y_2\}), (\{x_2, x_3, x_4\}, \{y_3, y_4\}), (\{x_4, x_5\}, \{y_4, y_5\})\}$$



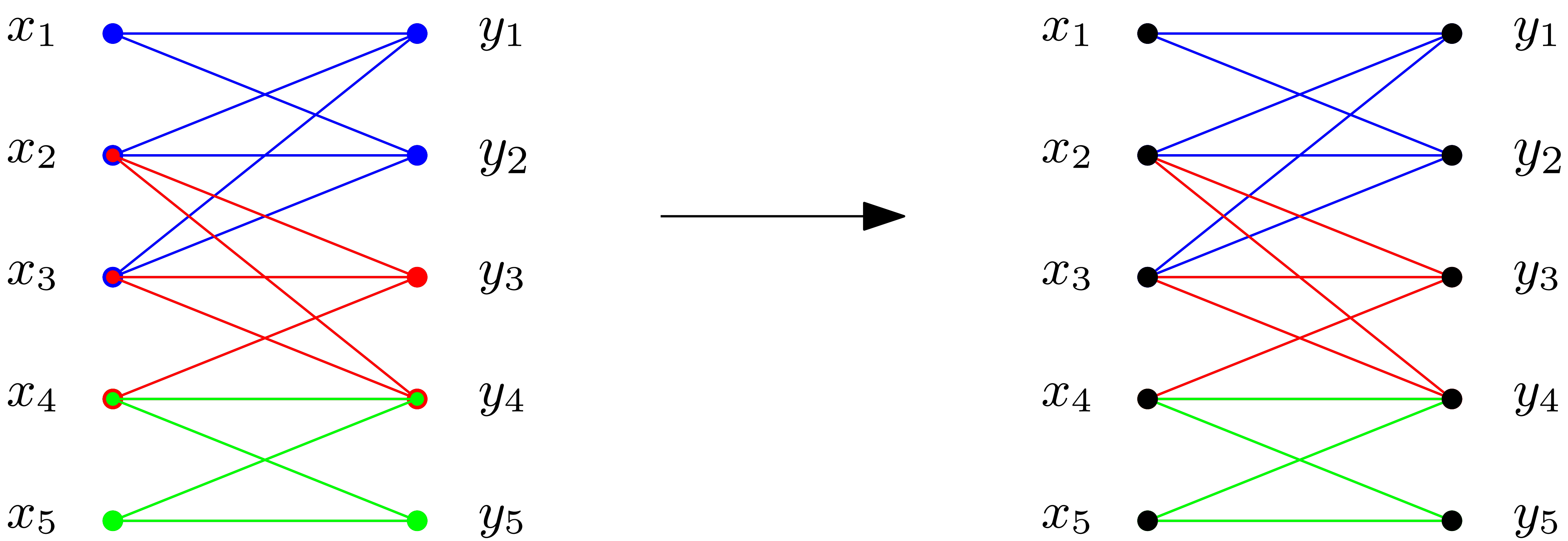
Chromatic Characterization

Let G be a bipartite graph.

Edge Colouring. No two monochromatic edges are independent.

Lemma. The edges of G can be covered with k bicliques if and only if the edges of G can be coloured with k colours.

$$\{(\{x_1, x_2, x_3\}, \{y_1, y_2\}), (\{x_2, x_3, x_4\}, \{y_3, y_4\}), (\{x_4, x_5\}, \{y_4, y_5\})$$
$$\{\{(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2), (x_3, y_1), (x_3, y_2)\}, ...\}$$



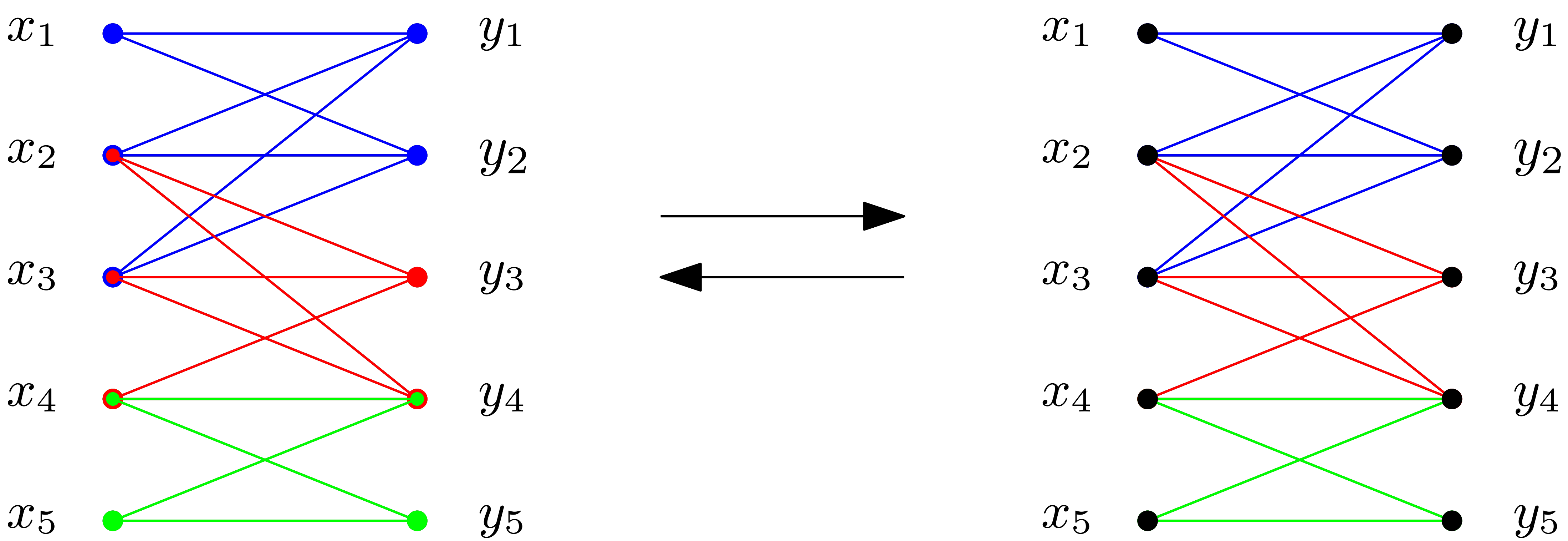
Chromatic Characterization

Let G be a bipartite graph.

Edge Colouring. No two monochromatic edges are independent.

Lemma. The edges of G can be covered with k bicliques if and only if the edges of G can be coloured with k colours.

$$\{(\{x_1, x_2, x_3\}, \{y_1, y_2\}), (\{x_2, x_3, x_4\}, \{y_3, y_4\}), (\{x_4, x_5\}, \{y_4, y_5\})$$
$$\{\{(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2), (x_3, y_1), (x_3, y_2)\}, \ldots\}$$



Colouring the Kernel Using Backtracking

Colour the uncovered edges of the kernel.

Let S be a maximal set of pairwise independent edges. Each edge in S is assigned a unique colour at the begining of the algorithm, and it is never recoloured.

Backtracking step. Pick an uncoloured edge e. Recursively assign to e each valid colour.

Branch on the most constrained edge (minimum number of valid colours).

Prune the search using the best-so-far solution.

Reduction + Backtracking Results

Running Time: less than 5 minutes.

Total Running Time: less than 15 minutes.

Dataset	LB	RC	BC	TC	GC
americas_small	173	159	19	178	220
americas_large	390	371	24	395	422

Fixed-parameter Tractability

Let $\langle G, k\rangle$ be an instance of **BC**.

Fixed-parameter Tractability

Let $\langle G, k \rangle$ be an instance of **BC**.

Rule*. If G has two vertices u, v that have the same neighbourhood, remove v from G.

Lemma. Rule* is safe.

Fixed-parameter Tractability

Let $\langle G, k \rangle$ be an instance of **BC**.

Rule*. If G has two vertices u, v that have the same neighbourhood, remove v from G.

Lemma. Rule* is safe.

Definition. The kernel $ker(G)$ of a bipartite graph G is the graph obtained from G by exhaustively applying Rule*.

Theorem. The kernel $ker(G)$ has at most 2^{k+1} vertices, or otherwise $\langle G, k \rangle$ has no solution.

Corollary. **BC** is fixed-parameter tractable with respect to parameter k (that is, there exists an $O(f(k) \cdot n^{O(1)})$ algorithm for the problem).

Thank You