CS 257: Numerical Methods Spring 2006 **February 7, 2006**

Q1 Let $A > 0$ and $g(x) = 2x - Ax^2$.

1. Show that if fixed-point iteration converges to a nonzero limit, then the limit is $x = 1/A$.
2. Find an interval about $1/A$ for which fixed-point iteration converges, provided x_0 is in that interval

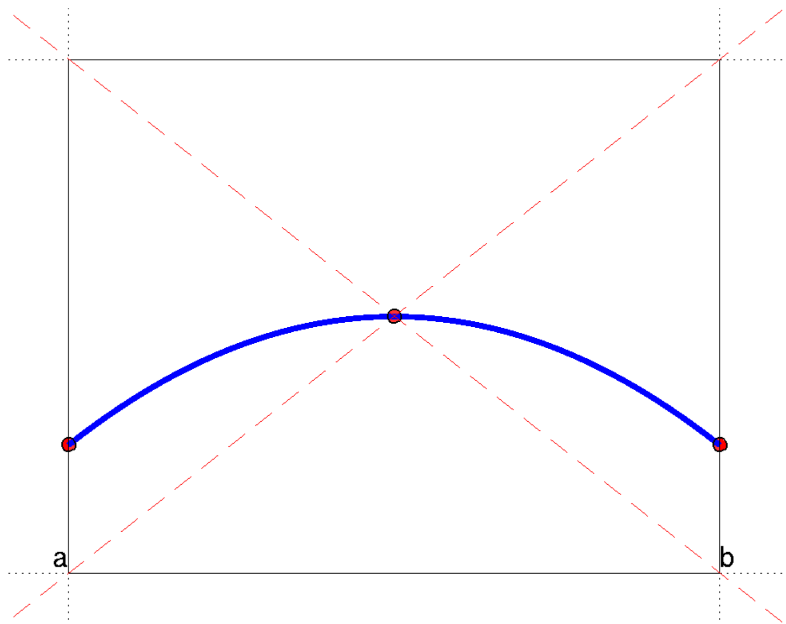
SOLUTION: For the first part, if fixed-point iteration converges, then $x = g(x)$. So $x = 2x - Ax^2$. Since $x \neq 0$, this gives $1 = 2 - Ax$ and $x = 1/A$. For the second part, we need to show a couple things: $g(x) \in [a, b]$ and $g'(x) \leq k < 1$ for $x \in [a, b]$ for some $[a, b]$. First find when g' is bounded by 1:

$$
\begin{aligned}
&|g'(x)| \leq \\
\Rightarrow &|2x - Ax^2| \leq 1 \\
\Rightarrow &-1 \leq 2x - Ax^2 \\
\text{and} \\
&2x - Ax^2 \leq 1 \\
\Rightarrow &\frac{1}{2A} \leq x \\
\text{and} \\
&x \leq \frac{3}{2A}
\end{aligned}
$$

Also, $g(x)$ attains a maximum when $g'(x) = 0$, which is at $x = 1/A$ (consequently). From this, we have

$$
\begin{aligned}
g(\frac{1}{A}) &= \frac{1}{A} \\
g(\frac{1}{2A}) &= \frac{3}{4A} \\
g(\frac{3}{2A}) &= \frac{3}{4A}
\end{aligned}
$$

All of the hypothesis are satisfied for any interval about $1/A$ inside of $[\frac{1}{2A}, \frac{3}{2A}]$.


a
b

Q2 Determine the number of iterations necessary to solve $f(x) = x^3 + x - 4 = 0$ with accuracy 10^{-3} in the interval $[a, b] = [1, 4]$. [*Note:* $\log_{10}(3) \approx 0.4771$ and $\log_{10}(2) \approx 0.3010$] **SOLUTION:** The function only tells us if we have a root in this interval:

$$\begin{aligned} f(1) &= -4 < 0 \\ f(4) &= 64 > 0 \qquad \Rightarrow \text{ bracket!} \end{aligned}$$

From the lecture, we know

$$|x_n - x| < \frac{b-a}{2^n}$$

So, we need

$$\begin{aligned} \frac{b-a}{2^n} &= \frac{4-1}{2^n} < 10^{-3} \\ &\Rightarrow \frac{3 \times 10^3}{<} 2^n \\ &\Rightarrow \frac{\log(3) + 3}{\log(2)} < n \\ &\Rightarrow \frac{.5 + 3}{.3} < \frac{\log(3) + 3}{\log(2)} < n \\ &\Rightarrow 11\frac{2}{3} \leq n \\ &\Rightarrow 12 \leq n \end{aligned}$$