Lecture 21

Splines

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Today:

Objectives

- automate Newton interpolation
- identify another problem with interpolation
- motivate a fix: splines

Material

- Section 10.3

Newton Polynomials

- To evaluate L_i^{n-1} we need to do a $\mathcal{O}(n^2)$ multiplications and additions. Furthermore, the computation may be suceptible to round off.
- ideally, we'd like to use nested iteration:

$$p_n(x) = c_1 + (x - x_1)(c_2 + (x - x_2)(c_3 + \dots))$$

- We can do this efficiently with Newton Polynomials

Newton Polynomials

- Newton Polynomials are of the form

$$p_n(x) = c_1 + c_2(x - x_1) + c_3(x - x_1)(x - x_2) + c_4(x - x_1)(x - x_2)(x - x_3) + \dots$$

- The basis used is thus

function	order
1	0
$x - x_1$	1
$(x - x_1)(x - x_2)$	2
$(x - x_1)(x - x_2)(x - x_3)$	3

- More stable that monomials
- More computationally efficient (nested iteration) than using Lagrange and shifted monomials

Newton Polynomials using Divided Differences

Using *divided differences*:

$$f[x_i, x_j] = \frac{f(x_j) - f(x_i)}{x_j - x_i}$$

we get the coefficients easily:

$$\begin{aligned} c_1 &= f[x_1] \\ c_2 &= f[x_1, x_2] \\ c_3 &= \frac{f[x_1, x_2] - f[x_2, x_3]}{x_3 - x_1} \\ &= f[x_1, x_2, x_3] \end{aligned}$$

Newton Polynomials using Divided Differences
example: long way

Example

For the data

x	1	-4	0
y	3	13	-23

Find the 2nd order interpolatin polynomial using Newton.

We know

$$p_2(x) = c_1 + c_2(x - x_1) + c_3(x - x_1)(x - x_2)$$

And that

$$c_1 = f[x_1] = f[1] = f(1) = 3$$

$$c_2 = f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{13 - 3}{-4 - 1} = -2$$

$$c_3 = f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$$

$$\frac{-23-13}{0--4} - \frac{13-3}{-4-1}$$

Newton Polynomials using Divided Differences

example: long way

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For the data

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Find the 2nd order interpolatin polynomial using Newton.

And

$$c_3 = f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$$

$$= \frac{\frac{-23-13}{0--4} - \frac{13-3}{-4-1}}{0-1}$$

$$= \frac{-9+2}{-1} = 7$$

So

$$p_2(x) = 3 - 2(x-1) + 7(x-1)(x+4)$$

Divided Differences

the easy way: tables

We can compute the divided differences much easier using tables. To construct the divided difference talbe for $f(x)$ for the $x_1, \ldots, x_4$

x	$f[\cdot]$	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot]$
x_1	$f[x_1]$			
		$f[x_1, x_2]$		
x_2	$f[x_2]$		$f[x_1, x_2, x_3]$	
		$f[x_2, x_3]$		$f[x_1, x_2, x_3, x_4]$
x_3	$f[x_3]$		$f[x_1, x_2, x_3]$	
		$f[x_3, x_4]$		
x_4	$f[x_4]$			

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Divided Differences

the easy way: example

Construct the divided differences talbe for the data

x	1	$\frac{3}{2}$	0	2
y	3	$\frac{13}{4}$	3	$\frac{5}{3}$

and construct the largest order interpolating polynomial.

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The coeffiecients are readily available and we arrive at

$$p_3(x) = 3 + \frac{1}{2}(x-1) + \frac{1}{3}(x-1)(x-\frac{3}{2}) - 2(x-1)(x-\frac{3}{2})$$