Lecture 18

Newton Interpolation

L. Olson

Department of Computer Science
University of Illinois at Urbana-Champaign

Slides based on slides from Numerical Methods with Matlab by Gerald Recktenwald

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Today:

Objectives

- look at two Examples of Lagrange interpolation
- introduce Newton interpolation
- highlight why Newton is efficient

Material

- Section 10.2

Lagrange

Example

Find the interpolating polynomial of least degree using a Lagrange basis that interpolates

x	1.4	1.25
y	3.7	3.9

Directly

$$
\begin{aligned}
p_1(x) &= \left(\frac{x-1.25}{1.4-1.25}\right)3.7 + \left(\frac{x-1.4}{1.25-1.4}\right)3.9 \\
&= 3.7 + \left(\frac{3.9-3.7}{1.25-1.4}\right)(x-1.4) \\
&= 3.7 - \frac{4}{3}(x-1.4)
\end{aligned}
$$

Lagrange

Example

Write the Lagrange basis functions for

x	$\frac{1}{3}$	$\frac{1}{4}$	1
y	2	-1	7

Directly

$$L_1^2(x) = \frac{(x-\frac{1}{4})(x-1)}{(\frac{1}{3}-\frac{1}{4})(\frac{1}{3}-1)}$$

$$L_2^2(x) = \frac{(x-\frac{1}{3})(x-1)}{(\frac{1}{4}-\frac{1}{3})(\frac{1}{4}-1)}$$

$$L_3^2(x) = \frac{(x-\frac{1}{3})(x-\frac{1}{4})}{(1-\frac{1}{3})(1-\frac{1}{4})}$$

Newton Polynomials

- To evaluate L_i^{n-1} we need to do a $\mathcal{O}(n^2)$ multiplications and additions. Furthermore, the computation may be suceptible to round off.
- ideally, we'd like to use nested iteration:

$$p_n(x) = c_1 + (x - x_1)(c_2 + (x - x_2)(c_3 + \dots))$$

- We can do this efficiently with Newton Polynomials

Newton Polynomials

- Newton Polynomials are of the form

$$p_n(x) = c_1 + c_2(x - x_1) + c_3(x - x_1)(x - x_2) + c_4(x - x_1)(x - x_2)(x - x_3) + \dots$$

- The basis used is thus

function	order
1	0
$x - x_1$	1
$(x - x_1)(x - x_2)$	2
$(x - x_1)(x - x_2)(x - x_3)$	3

- More stable that monomials
- More computationally efficient (nested iteration) than using Lagrange and shifted monomials

Newton Polynomials using Divided Differences

Consider the data

x_1	x_2	x_3
y_1	y_2	y_3

We want to find c_1, c_2, and c_3 in the following polynomial so that it fits the data:

$$p_2(x) = c_1 + c_2(x - x_1) + c_3(x - x_1)(x - x_2)$$

Matching the data gives three equations to determine our three unknowns c_i:

at x_1: $y_1 = c_1 + 0 + 0$
at x_2: $y_2 = c_1 + c_2(x_2 - x_1) + 0$
at x_3: $y_3 = c_1 + c_2(x_3 - x_1) + c_3(x_3 - x_1)(x_3 - x_2)$

Newton Polynomials using Divided Differences

Or in matrix form:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & x_2 - x_1 & 0 \\ 1 & x_3 - x_1 & (x_3 - x_1)(x_3 - x_2) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$\Rightarrow$ lower triangular
$\Rightarrow$ only $\mathcal{O}(n^2)$ operations

Newton Polynomials using Divided Differences

Using Forward Substitution to solve this lower triangular system yeilds:

$$
\begin{aligned}
c_1 &= y_1 = f(x_1) \\
c_2 &= \frac{y_2 - c_1}{x_2 - x_1} \\
&= \frac{f(x_2) - f(x_1)}{x_2 - x_1} \\
c_3 &= \frac{y_3 - c_1 - (x_3 - x_1)c_2}{(x_3 - x_2)(x_3 - x_1)} \\
&= \frac{f(x_3) - f(x_1) - (x_3 - x_1)\frac{f(x_2) - f(x_1)}{x_2 - x_1}}{(x_3 - x_2)(x_3 - x_1)} \\
&= \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_3) - f(x_2)}{x_3 - x_2}}{x_3 - x_1}
\end{aligned}
$$

Newton Polynomials using Divided Differences

From this we see a pattern. That there are many terms of the form

$$\frac{f(x_j) - f(x_i)}{x_j - x_i}$$

These are called *divided differences* and are denoted with square brackets:

$$f[x_i, x_j] = \frac{f(x_j) - f(x_i)}{x_j - x_i}$$

Applying this to our results:

$$\begin{aligned}
c_1 &= f[x_1] \\
c_2 &= f[x_1, x_2] \\
c_3 &= \frac{f[x_1, x_2] - f[x_2, x_3]}{x_3 - x_1} \\
&= f[x_1, x_2, x_3]
\end{aligned}$$

Newton Polynomials using Divided Differences

example: long way

Example

For the data

x	1	-4	0
y	3	13	-23

Find the 2nd order interpolatin polynomial using Newton.

We know

$$p_2(x) = c_1 + c_2(x - x_1) + c_3(x - x_1)(x - x_2)$$

And that

$$c_1 = f[x_1] = f[1] = f(1) = 3$$

$$c_2 = f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{13 - 3}{-4 - 1} = -2$$

$$c_3 = f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$$

$$\frac{-23-13}{0--4} - \frac{13-3}{-4-1}$$

Newton Polynomials using Divided Differences

example: long way

Example

For the data

x	1	-4	0
y	3	13	-23

Find the 2nd order interpolatin polynomial using Newton.

And

$$c_3 = f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$$

$$= \frac{\frac{-23-13}{0--4} - \frac{13-3}{-4-1}}{0-1}$$

$$= \frac{-9+2}{-1} = 7$$

So

$$p_2(x) = 3 - 2(x-1) + 7(x-1)(x+4)$$

Divided Differences

Recursive Property

$$f[x_1, \ldots, x_k] = \frac{f[x_2, \ldots, x_k] - f[x_1, \ldots, x_{k-1}]}{x_k - x_1}$$

With the first two defined by

$$f[x_i] = f(x_i)$$

$$f[x_i, x_j] = \frac{f[x_j] - f[x_i]}{x_j - x_i}$$

Divided Differences

Invariance Theorem

$f[x_1, \ldots, x_k]$ is invariant under all permutations of the arguments $x_1, \ldots, x_k$

This says that we can also write

$$f[x_i, \ldots, x_j] = \frac{f[x_{i+1}, \ldots, x_j] - f[x_i, \ldots, x_{j-1}]}{x_j - x_i}$$

Divided Differences
the easy way: tables

We can compute the divided differences much easier using tables. To construct the divided difference talbe for $f(x)$ for the $x_1, \ldots, x_4$

x	$f[\cdot]$	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot]$
x_1	$f[x_1]$			
		$f[x_1, x_2]$		
x_2	$f[x_2]$		$f[x_1, x_2, x_3]$	
		$f[x_2, x_3]$		$f[x_1, x_2, x_3, x_4]$
x_3	$f[x_3]$		$f[x_1, x_2, x_3]$	
		$f[x_3, x_4]$		
x_4	$f[x_4]$			

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x_1	$f[x_1]$			
		$f[x_1, x_2]$		
x_2	$f[x_2]$		$f[x_1, x_2, x_3]$	
		$f[x_2, x_3]$		$f[x_1, x_2, x_3, x_4]$
x_3	$f[x_3]$		$f[x_1, x_2, x_3]$	
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		$f[x_1, x_2]$		
x_2	$f[x_2]$		$f[x_1, x_2, x_3]$	
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Divided Differences

the easy way: example

Construct the divided differences talbe for the data

x	1	$\frac{3}{2}$	0	2
y	3	$\frac{13}{4}$	3	$\frac{5}{3}$

and construct the largest order interpolating polynomial.

We can compute the divided differences much easier using tables. To construct the divided difference talbe for $f(x)$ for the $x_1, \ldots, x_4$

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1	3			
		$\frac{1}{2}$		
$\frac{3}{2}$	$\frac{13}{4}$		$\frac{1}{3}$	
		$\frac{1}{6}$		-2
0	3		$-\frac{5}{3}$	
		$-\frac{2}{3}$		
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The coeffiecients are readily available and we arrive at

$$p_3(x) = 3 + \frac{1}{2}(x-1) + \frac{1}{3}(x-1)(x-\frac{3}{2}) - 2(x-1)(x-\frac{3}{2})$$

Projects

Two project ideas from this material:

1. Neville's Algorithm
   - fast algorithm for Newton interpolation
   - the project would involve creating a neville function
   - other aspects might include performace timing
2. interpolation in 2-D?
   - the project would involve looking at Lagrange interpolation in 2-D
   - highlight when you can use it and give some examples

Matlab

Two very good scripts are included with your book. Take a look at each of them to

- reconstruct example 10.10 with `compInterp`
- reconstruct example 10.11 with `demoWiggle`