NAME: ____

AM 034

Brown University — **Fall 2004**

FinalExam — **Wednesday December 15, 2004**

No computers, calculators, books, notes, or crib sheet allowed. Write your **name** on each sheet of paper and start each new problem on a new page. For full credit, **show** all work.

(20 pts.) **1.** Consider a metal rod of length 2 and thermal diffusivity 1. The ends of the rod are held at a constant temperature of 1° and the initial temperature distribution in the rod is given by

$$f(x) = \begin{cases} x, & 0 \le x \le 1; \\ 2-x, & 1 \le x \le 2. \end{cases}$$

(i) Find $u(x,t)$, explicitly.

(ii) Sketch $u(x,0)$ and sketch (based on your intuition) $u(x,t)$ at a few selected times.

Solution: First make the steady state solution: $v(x) = 1$ and notice that the function $w(x,t) = u(x,t) - v(x)$ satisfies homogeneous boundary conditions: $w(0,t) = w(2,t) = 0$. Then

$$w(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t/4} \sin(n\pi x/2).$$

Now for the initial conditions. Notice that $f(x) - 1$ is $x - 1$ and $1 - x$ respectively. Thus the integral

$$\begin{aligned} F(x) &= \int (x-1)\sin(n\pi x/2)\,dx \\ &= -\frac{2}{n\pi}(x-1)\cos(n\pi x/2) + \int \frac{2}{n\pi}\cos(n\pi x/2) \\ &= -\frac{2}{n\pi}(x-1)\cos(n\pi x/2) + \frac{4}{n^2\pi^2}\sin(n\pi x/2), \end{aligned}$$

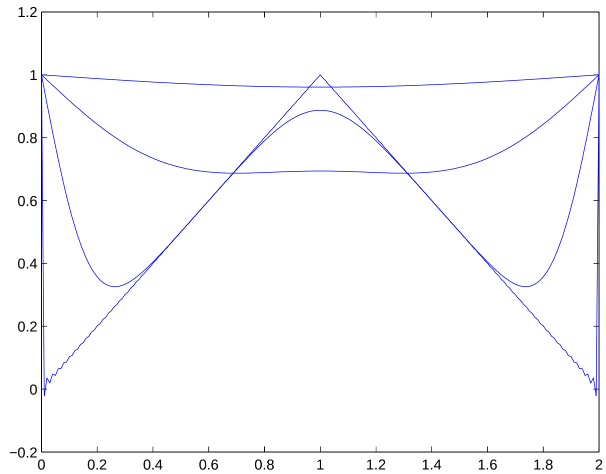
serves for both pieces. Thus

$$\begin{aligned} c_n =& \frac{2}{2}\left(\int_0^1 (x-1)\sin(n\pi x/2)\,dx + \int_1^2 (1-x)\sin(n\pi x/2)\,dx\right) \\ &= F|_0^1 - F|_1^2 \\ &= F(1) - F(0) + F(1) - F(2) \\ &= \frac{8}{n^2\pi^2}\sin(n\pi/2) - \frac{2}{n\pi} + \frac{2}{n\pi}\cos(n\pi). \end{aligned}$$

This is fine. Thus the solution is

$$u(x,t) = 1 + \sum_{n=1}^{\infty}\left(\frac{8}{n^2\pi^2}\sin(n\pi/2) - \frac{2}{n\pi} + \frac{2}{n\pi}\cos(n\pi)\right) e^{n^2\pi^2 t/4}\sin(n\pi x/2).$$

See figure.



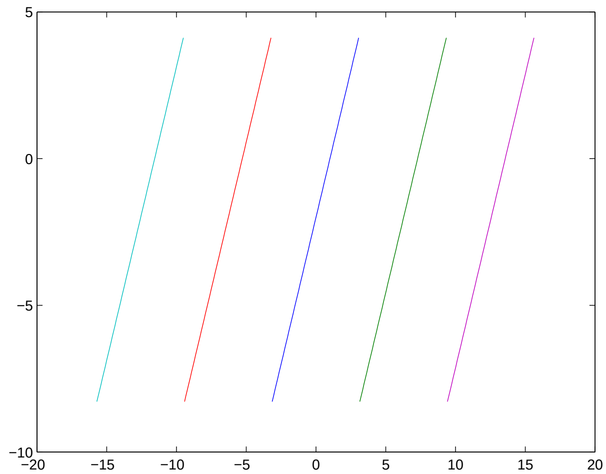
(20 pts.) **2.** Consider the 2π periodic function

$$f(x) = 2x - 2, \qquad x \in [-\pi, \pi].$$

(i) Sketch the function and determine if it is odd, even, or neither.

(ii) Find the Fourier series representation.

(iii) True or False: The partial sums $f_N(x)$ will converge in the 'maximum norm' (i.e. the pointwise maximum of the absolute value). Explain in one sentence why.



Solution: Neither odd nor even...see figure. The fourier series needs

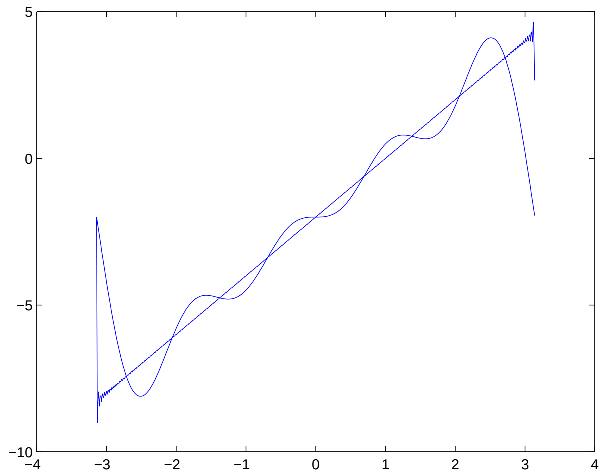
$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(2x-2)\sin nx\,dx \\ &= \frac{1}{\pi}\left(-\frac{2x-2}{n}\cos(nx)+\frac{2}{n^2}\sin(nx)\right)\Big|_{-\pi}^{\pi} \\ &= \frac{1}{n\pi}\left(-4\pi\cos(n\pi)\right) \\ &= \frac{4(-1)^{n+1}}{n} \\ a_n &= \int_{-\pi}^{\pi}(2x-2)\cos nx\,dx \\ &= 0. \\ a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi}2x-2\,dx \\ &= \frac{1}{\pi}\left(\pi^2-2\pi-\pi^2+2\pi\right) \\ &= -4 \end{aligned}$$

Thus

$$f(x) = -2+\sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{n}\sin(nx).$$

See figure for $N = 400$ and $N = 4$, for example.
False. Since the term is like $\frac{1}{n}$, we won't converge. Here, $p+1 = 1$, so that the error looks like $1/N^p = 1$. This is due to the fact that we are approximating a jump with continuous functions. See figure.

(15 pts.) **3.** Determine if $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ is deficient (*hint*: show $p(\lambda) = -(\lambda-1)^2(\lambda-5)$).



Solution:

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix}$$
$$= (2-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 2 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 3-\lambda & 1 \end{vmatrix}$$
$$= (2-\lambda)(\lambda^2-5\lambda+4)+2\lambda-2-1+\lambda$$
$$= -\lambda^3+7\lambda^2-11\lambda+5$$
$$= -(\lambda-1)^2(\lambda-5).$$

Thus $\lambda = 1$ of multiplicity 2 and $\lambda = 5$ of multiplicity 1. It is sufficient to only check $\lambda = 1$. So

$$A - I = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

Thus, $\xi_1 + 2\xi_2 + \xi_3 = 0$. Pick $\xi_1 = \alpha$ and $\xi_2 = \beta$, which yields $\xi_3 = -\alpha - 2\beta$. So $\xi = \begin{bmatrix} \alpha & \beta & -\alpha-2\beta \end{bmatrix}^T = \alpha \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T + \beta \begin{bmatrix} 0 & 1 & -2 \end{bmatrix}^T$. Now the vectors $\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T$ and $\begin{bmatrix} 0 & 1 & -2 \end{bmatrix}^T$ are linearly independent, so the matrix in **not** deficient.

(15 pts.) **4.** Find the matrix exponential e^{At} for $A = \begin{bmatrix} 5 & 3 \\ -6 & -4 \end{bmatrix}$.

Solution: One way is to form $\Psi(t)\Psi^{-1}(0)$. $|A-\lambda I| = \begin{vmatrix} 5-\lambda & 3 \\ -6 & -4-\lambda \end{vmatrix} = \lambda^2+\lambda-2 = 0$. Thus $(\lambda+1)(\lambda-2) = 0$ and $\lambda_1 = -1$ and $\lambda_2 = 2$. The associated eigenvectors are $\begin{bmatrix} 1 & -2 \end{bmatrix}^T$ and $\begin{bmatrix} 1 & -1 \end{bmatrix}^T$. Then $\Psi(t) = \begin{bmatrix} e^{-t} & e^{2t} \\ -2e^{-t} & -e^{2t} \end{bmatrix}$ and $\Psi^{-1}(0) = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$. The exponential is then

$$\begin{aligned} e^{At} &= \Psi(t)\Psi^{-1}(0) \\ &= \begin{bmatrix} e^{-t} & e^{2t} \\ -2e^{-t} & -e^{2t} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -e^{-t}+2e^{2t} & -e^{-t}+e^{2t} \\ 2e^{-t}-2e^{2t} & 2e^{-t}-e^{2t} \end{bmatrix}. \end{aligned}$$

(15 pts.) **5.** Discuss the stability properties of $x' = y$, $y' = -6x - y - 3x^2$ at each of its equilibrium points.

Solution: The critical points are $(0,0)$ and $(-2,0)$. The Jacobian is

$$J(x,y) = \begin{bmatrix} 0 & 1 \\ -6-6x & -1 \end{bmatrix}.$$

So for $(0,0)$, the matrix is $A = \begin{bmatrix} 0 & 1 \\ -6 & -1 \end{bmatrix}$, which has eigenvalues $\lambda = -0.5 \pm \textit{imaginary}$, so it is asymptotically stable. Likewise at $(-2,0)$, the matrix is $A = \begin{bmatrix} 0 & 1 \\ 6 & -1 \end{bmatrix}$. The eigs are $\lambda_1 = 2$ and $\lambda_2 = -3$, so the point is a saddle point. These stabilities are completely determined from the linearized system.

(15 pts.) **6.** Consider the system $x' = y$, $y' = -x - y$.

(i) Can we use the Liapunov function $V(x,y) = x^2 + y^2$ to determine stability of the origin? (show)

(i) Can we use the Liapunov function $V(x,y) = 3x^2 + 2xy + 2y^2$ to determine stability of the origin? (show)

Solution: for (i),

$$\begin{aligned}\frac{DV}{Dt} &= 2xx' + 2yy' \\ &= 2xy - 2xy - 2y^2 \\ &= -2y^2.\end{aligned}$$

$V(x, y)$ is positive definite since $4ac - b^2 = 4 * 2 * 2 > 0$, but V' is only negative semi-definite...which means we can only say that the point is stable. Using the function in part (ii), V is again positive definite since $4ac - b^2 = 4 * 3 * 2 - 1 > 0$ and we see that

$$\begin{aligned}\frac{DV}{Dt} &= 6xx' + 2xy' + 2x'y + 4yy' \\ &= (6x + 2y)x' + (2x + 4y)y' \\ &= 6xy + 2y^2 + -2x^2 - 2xy - 4xy - 4y^2 \\ &= -2x^2 - 2y^2.\end{aligned}$$

This **is** negative definite, so it tells us a little more: that the origin is asymptotically stable.