**NAME**: ______________________________________________

**AM 034**

**Brown University** **Fall 2004**
**Exam #1Solutions** **Friday October 8, 2004**

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**No** computers, calculators, books, notes, or crib sheet allowed. Write your **name** on each sheet of paper and start each new problem on a new page. For full credit, **show** all work.

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(15 pts.) **1.** Does $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$ have (choose one)

(i) no solutions?

(ii) a unique solution?

(iii) infinitely many solutions?

**Solution:** Begin by augmenting:

$$\begin{bmatrix} 2 & -1 & 1 & | & 3 \\ -1 & 2 & 1 & | & 0 \\ 1 & 1 & 2 & | & 3 \end{bmatrix} \overset{\substack{R_2+R_3 \to R_3 \\ 1/2R_2 \to R_1}}{\longrightarrow} \begin{bmatrix} 1 & -1/2 & 1/2 & | & 3/2 \\ -1 & 2 & 1 & | & 0 \\ 0 & 3 & 3 & | & 3 \end{bmatrix}$$

$$\overset{-1 \cdot R_1 + R_2 \to R_2}{\longrightarrow} \begin{bmatrix} 1 & -1/2 & 1/2 & | & 3/2 \\ 0 & 3/2 & 3/2 & | & 3/2 \\ 0 & 3 & 3 & | & 3 \end{bmatrix}$$

$$\overset{\substack{-1 \cdot R_2 + R_3 \to R_3 \\ 2/3R_2 \to R_2}}{\longrightarrow} \begin{bmatrix} 1 & -1/2 & 1/2 & | & 3/2 \\ 0 & 1 & 1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Since **anything** can satisfy the last equation, the system is undertermined and we have (*iii*), infinitely many solutions.

(15 pts.) **2.** Write

$$2y''(t) + 2y'(t) + y(t) = 0,$$

as a first order system: $\mathbf{x}' = A\mathbf{x}$. [**Bonus** (3pts): Classify the system (nodal source/sink, spiral source/sink, saddle, center, star, deficient node)]

**Solution:** Let

$$\begin{cases} x_1 &= y \\ x_2 &= y' \end{cases} \Rightarrow \begin{cases} x_1' &= y' \\ x_2' &= y'' \end{cases}$$
$$\Rightarrow \begin{cases} x_1' &= x_2 \\ x_2' &= 1/2(-2y' - y) \end{cases}$$
$$\Rightarrow \begin{cases} x_1' &= x_2 \\ x_2' &= -1/2x_1 - x_2 \end{cases}$$

So that $\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -1/2 & -1 \end{bmatrix} \mathbf{x}$. [**Bonus**: The eigenvalues are easily computed as $\lambda = -1/2 \pm 1/2i$, so we have a spiral sink.]

(10 pts.) **3.** Define "degenerate matrix" (2-3 sentences).

**Solution:** A degenerate matrix is a matrix with nonsimple eigenvalues that are not associeated with a full set of corresponding eigenvectors. That is, there is at least one eigenvalue of multiplicity $m$ greater than one with less than $m$ associated eigenvectors.

(30 pts.) **4.** For

$$\begin{aligned} x_1' &= 3x_1 - 2x_2 \\ x_2' &= 2x_1 - 2x_2, \end{aligned}$$

with $x_1(0) = 1$ and $x_2(0) = 1$,

(i) Find the general solution

(ii) Find the unique solution that satisfies the IVP using $e^{At}$.

(iii) Classify the system (nodal source/sink, spiral source/sink, saddle, center, star, deficient node)

(iv) Sketch a simple phase portrait indicating relavant features.

**Solution:** The matrix in $\mathbf{x}' = A\mathbf{x}$ is $A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$.

**(i)** Find $(\lambda, \xi)$ pairs: $|A - \lambda I| = \begin{vmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = 0$. So $\lambda_{1,2} = -1, 2$. The associated eigenvectors are then $\xi_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\xi_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. From this, the **general** solution is

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} + c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t}.$$

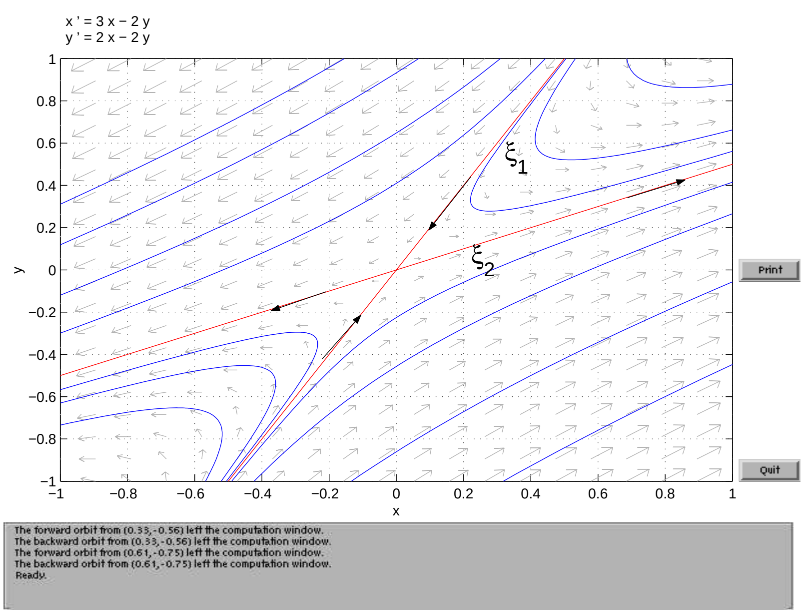
**(ii)** First find $e^{At}$:

$$\begin{aligned} e^{At} &= Te^{Dt}T^{-1} \\ &= \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} \left(-\frac{1}{3}\right) \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \\ &= \left(-\frac{1}{3}\right) \begin{bmatrix} e^{-t} - 4e^{2t} & -2e^{-t} + 2e^{2t} \\ 2e^{-t} - 2e^{2t} & -4e^{-t} + e^{2t} \end{bmatrix} \end{aligned}$$

The **unique** solution is

$$\begin{aligned} \mathbf{x}(t) &= \Phi(t)\mathbf{x}^0 \\ &= e^{At}\mathbf{x}^0 \\ &= \left(-\frac{1}{3}\right) \begin{bmatrix} e^{-t} - 4e^{2t} & -2e^{-t} + 2e^{2t} \\ 2e^{-t} - 2e^{2t} & -4e^{-t} + e^{2t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad = \left(\frac{1}{3}\right) \begin{bmatrix} e^{-t} + 2e^{2t} \\ 2e^{-t} + e^{2t}. \end{bmatrix} \end{aligned}$$

**(iii)** Since $\lambda_1 < 0 < \lambda_2$, it is a saddle node.




**(iv)**

(30 pts.) **5.** For $A = \begin{bmatrix} -2 & -1 & 1 \\ 2 & 1 & 0 \\ 3 & 1 & -1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, if possible,

(i) find $A^{-1}$

(ii) find $det(A)$

(iii) find $\mathbf{x}^T A$

**Solution:** Augment the matrix $[A|I]$:

**(i)**

$$\left[\begin{array}{ccc|ccc} -2 & -1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1+R_3 \to R_3]{R_1+R_2 \to R_2} \left[\begin{array}{ccc|ccc} -2 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\substack{R_2 \to R_3 \\ R_3 \to R_1}]{R_1 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ -2 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[-1 \cdot R_2 \to R_2]{2 \cdot R_1 + R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & -3 & 0 & -2 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 + R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -2 & 1 & -2 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & -2 \\ 1 & 1 & 0 \end{bmatrix}$.

**(ii)**

$$det(A) = -2 \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix} + 1 \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} + 0\,|\cdot| = -1$$

**(iii)**

$$\begin{aligned} \mathbf{x}^T A &= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -2 & -1 & 1 \\ 2 & 1 & 0 \\ 3 & 1 & -1 \end{bmatrix} \\ &= (1 \times 3)(3 \times 3) \\ &= \begin{bmatrix} (-2+4+9) & (-1+2+3) & (10-3) \end{bmatrix} \\ &= \begin{bmatrix} 11 & 4 & -2 \end{bmatrix} \\ &= (1 \times 3) \end{aligned}$$