NAME: ______________________________________

AM 033 — Applied Mathematics - I

Brown University **Fall 2003**

Midterm Exam #2 **Friday, November 7, 2003**

Answer **all** 4 questions on separate paper (provided). Partial credit will be given for correct steps. Begin each solution write-up on a blank sheet of paper. Staple all sheets, including this cover sheet, and hand in. Crib sheets, calculators, and books are not permitted.

[**25 pts.**] **1.** Choose the most appropriate answer for each of the following questions:

[**5 pts.**] **(a)** The spring system modeled by

$$y'' + y' + 9.25y = 0$$

is

(i) underdamped

(ii) critically damped

(iii) overdamped

(iv) the system is not damped

[**5 pts.**] **(b)** The ODE

$$9y'' + 8e^{-t}y' + (3.14159 + t)y = 0.5$$

is

(i) linear, homogeneous with constant coefficients

(ii) linear, nonhomogeneous with constant coefficients

(iii) linear, nonhomogeneous

(iv) nonlinear, homogeneous

[**5 pts.**] **(c)** If $y_1(t)$ and $y_2(t)$ are both solutions to the ODE $y''+3y'+7y = f(t)$, then $y_1(t) + 3y_2(t)$ is also a solution.

(i) True (ii) **[False]**

[**5 pts.**] **(d)** Do the functions $y_1(x) = x^2$ and $y_2(x) = \frac{1}{x}$ form a fundamental solution set to the ODE $x^2y'' - 2y = 0$?

(i) **[Yes]** (ii) No

[**5 pts.**] **(e)** For which frequency, ω, does the following ODE exhibit resonnance?

$$y'' + 9y = A \sin \omega t$$

(i) $\sqrt{3}$

(ii) **[3]**

(ii) 9

(ii) $\frac{\sqrt{32}}{2}$

[**30 pts.**] **2.** Find the general solution to

[**15 pts.**] **(a)** $y'' + 2y' + 5y = 0$

[**15 pts.**] **(b)** $y'' - 6y' + 9y = 0$

Solution:

(a)

The characteristic equation $r^2 + 2r + 5 = 0$ yields complex roots

$$r = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2\imath.$$

Thus

$$y(t) = e^{-t}(c_1 \cos(2t) + c_2 \sin(2t)).$$

(b)

The characteristic equation $r^2 - 6r + 9 = 0$ yields the repeated root $r = 3$. Thus

$$y(t) = (c_1 + c_2 t)e^{3t}.$$

[30 pts.] 3. Find the form of a particular solution $y_p(t)$ to the following ODEs to be used in the method of undetermined coefficients. **Do not solve for the coefficients.**

[15 pts.] (a) $y'' + 9y = \cos(3t)$

[15 pts.] (b) $y'' + 9y = t^2 \cos(t)$

Solution:
From problem 1 part (e) the complementary solution is

$$y_c = c_1 \cos(3t) + c_2 \sin(3t)$$

(a)

$$\Rightarrow \quad y_p = At\cos(3t) + Bt\sin(3t)$$

(b)

$$\Rightarrow \quad y_p = (a_2 t^2 + a_1 t + a_0)\cos(t) + (b_2 t^2 + b_1 t + b_0)\sin(t)$$

[15 pts.] 4. Find the general solution to the following ODE using the method of variation of parameters:

$$y'' - 5y' + 6y = 2e^t$$

Solution: The characteristic equation $r^2 - 5r - 6 = 0$ yields 2 real and distinct roots

$$\begin{aligned} r_{1,2} &= \frac{5 \pm \sqrt{25-24}}{2} \\ &= \frac{5}{2} \pm \frac{1}{2} \\ &= 2, 3. \end{aligned}$$

Thus, we have the complementary solution

$$y_c = c_1 e^{2t} + c_2 e^{3t}.$$

Then our two solutions are

$$y_1 = e^{2t} \quad \text{and} \quad y_2 = e^{3t}.$$

From this we find the variable parameters u_1 and u_2 in the particular solution $y_p = u_1(t)y_1 + u_2(t)y_2$. This solution is constrained by the following two equations

$$u_1' y_1 + u_2' y_2 = 0 \tag{1}$$

$$u_1' y_1' + u_2' y_2' = g, \tag{2}$$

where $g(t) = 2e^{3t}$ for this ODE. The solutions to this are

$$u_1 = -\int \frac{y_2 g}{W}\, dt \tag{i}$$

$$u_2 = \int \frac{y_1 g}{W}\, dt, \tag{ii}$$

where

$$\begin{aligned} W &= W(y_1, y_2)(t) \\ &= \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} \\ &= e^{5t} \end{aligned}$$

Solution continued: Substituting into (i) and (ii) we have

$$\begin{aligned} u_1 &= -\int \frac{e^{3t} \cdot 2e^{t}}{e^{5t}}\, dt \\ &= 2e^{-t} \end{aligned}$$

and

$$\begin{aligned} u_2 &= \int \frac{e^{2t} \cdot 2e^{t}}{e^{5t}}\, dt \\ &= -e^{-2t} \end{aligned}$$

Thus

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \\ &= 2e^{-t}e^{2t} - e^{-2t}e^{3t} \\ &= (2-1)\, e^{t} \\ &= e^{t}. \end{aligned}$$

Finally,

$$\begin{aligned} y(t) &= y_c + y_p \\ &= c_1 e^{2t} + c_2 e^{3t} + e^{t} \end{aligned}$$