AM 033 — Applied Mathematics - I

Brown University — **Fall 2003**

Midterm Exam #1 — **SOLUTIONS** — **Thursday, October 2, 2003**

[**20 pts.**] **1.** Consider the ODE

$$y' = \frac{y^2 + xy}{x^2}.$$

[**5 pts.**] **(a)** Is the ODE

	yes	no
first-order?	$\checkmark$	___
linear?	___	$\checkmark$
separable?	___	$\checkmark$
autonomous?	___	$\checkmark$
homogeneous of order zero?	$\checkmark$	___

[**15 pts.**] **(b)** Solve the ODE.

Solution. We introduce the new dependent variable $y = vx$. Then

$$y' = v'x + v = v^2 + v \qquad \Longrightarrow \qquad \frac{dv}{v^2} = \frac{dx}{x}.$$

Integration yields

$$v = -\frac{1}{\ln Cx} \qquad \Longrightarrow \qquad y(x) = -\frac{x}{\ln Cx},$$

since $v = y/x$.

If we consider the given equation as the **Bernoulli equation**

$$y' = \frac{1}{x}\, y + \frac{1}{x^2}\, y^2$$

and make the substitution $u = y^{-1}$ $(-y'y^{-2} = u')$ then we obtain

$$-y^2\, u' = \frac{1}{x}\, y + \frac{1}{x^2}\, y^2 \qquad \Longrightarrow \qquad u' = -\frac{1}{x}\, y^{-1} - \frac{1}{x^2} = -\frac{1}{x}\, u - \frac{1}{x^2}$$

which is a linear differential equation in u. Multiplication by x reduces the equation to the exact one:

$$\frac{d}{dx}\,[xu] = -\frac{1}{x} \qquad \Longrightarrow \qquad xu = -\ln Cx.$$

Hence $u = y^{-1} = -\ln Cx/x$ and we obtain the same solution.

[**25 pts.**] **2.** Consider the ODE

$$(2xy^3 + y^4)dx + (xy^3 - 2)dy = 0.$$

[10 pts.] **(a)** Find an integrating factor μ that makes this equation exact.

[15 pts.] **(b)** Find the general solution to the ODE. [**Note:** Solution may be left in implicit form.]

Solution. **(a)** We set $M(x, y) = 2xy^3 + y^4$ and $N(x, y) = xy^3 - 2$. Then taking partial derivatives, we obtain

$$M_y = 6xy^2 + 4y^3, \quad N_x = y^3 \qquad \Longrightarrow \qquad \frac{M_y - N_x}{M} = \frac{6xy^2 + 4y^3 - y^3}{2xy^3 + y^4} = \frac{3y^2(2x + y)}{y^3(2x + y)} = \frac{3}{y}.$$

Therefore there exists an integrating factor as a function on y:

$$\mu(y) = e^{-\int \frac{M_y - N_x}{M}\, dy} = y^{-3}.$$

(b) Multiplication by $\mu(y)$ leads to the exact equation

$$(2x + y)\, dx + (x - 2y^{-3})\, dy = 0.$$

So there exists a potential function $\psi(x, y)$ such that $\psi_x = 2x + y$ and $\psi_y = x - 2y^{-3}$. Integration yields

$$\psi(x, y) = x^2 + xy + y^{-2} \qquad \Longrightarrow \qquad \text{the general solution is } \psi(x, y) = C,$$

where C is an arbitrary constant.

[25 pts.] **3.** Consider the Bernoulli equation

$$y' = y + xy^2.$$

[20 pts.] **(a)** Find its general solution [**hint:** recall the integration by parts formula $\int u\, dv = uv - \int v\, du$].

[5 pts.] **(b)** Specify a solution that satisfies the initial condition: $y(0) = 1/2$.

Solution. We make the substitution $u = y^{-1}$ to obtain

$$-y^2 u' = y + xy^2 \qquad \Longrightarrow \qquad u' + u + x = 0$$

Using an integrating factor $\mu(x) = e^x$, we get the exact equation:

$$\frac{d}{dx}\,[e^x u] = -x\, e^x \qquad \Longrightarrow \qquad e^x u = -\int x\, e^x\, dx + C = -x\, e^x + e^x + C.$$

Hence

$$u(x) = y^{-1}(x) = 1 - x + C\, e^{-x}; \qquad C = 1.$$

[30 pts.] 4. There is a trout fishing hole, of about 1000 liters in size, located on the Upper Iowa River. The rate of water flowing into the hole is approximately 100 liters/min. The rate flowing out is the same. Suppose there is a fertilizer spill just up stream and contaminated water starts to flow into the fishing hole with a concentration of 0.2 kg/liter. Assume the water is well mixed and that the initial amount contamination in the hole is zero liters. [Note: Use $ln(2) \approx 0.7$, $ln(1/2) \approx -0.7$, and $e \approx 3$ for your final answers if you wish.]

[10 pts.] (a) Set up the ODE that describes the amount of fertilizer $Q(t)$ in the hole at time t. [**hint:** recall $dQ/dt =$ Rate in - Rate out].

[10 pts.] (b) The fish will stop biting after the water has 0.1 Kg/liter of contaminant. How long can I successfully fish (in minutes)?

[10 pts.] (c) The fishing hole is dammed after 10 minutes to stop the flow of water and contamination downstream. How much fertilizer (in kilograms) has flowed out of the fishing hole between $t = 0$ and $t = 10$?

Solution. (a) Rate in = 0.2 kg/liter × 100 liter/min = 20 kg/min. Rate out = Q/100 kg/liter × 100 liter/min = 0.1 Q kg/min. Therefore

$$\frac{dQ}{dt} = 20 - 0.1\,Q, \qquad Q(0) = 0.$$

(b) The linear equation can be solve with the aid of an integrating factor $\mu(x) = e^{t/10}$:

$$\frac{d}{dt}\left[e^{t/10}\,Q\right] = 20\,e^{t/10} \qquad \Longrightarrow \qquad Q(t) = 200 - 200\,e^{-t/10}.$$

Now we solve for t:

$$0.1 \times 1000 = 100 = 200 - 200\,e^{-t/10} \qquad \Longrightarrow \qquad \frac{1}{2} = e^{-t/10} \qquad \Longrightarrow \qquad t = 10\ln 2 \approx 7\,\text{min}.$$

(c)

$$\int_0^{10} \text{Rate out}\,dt = 0.1\int_0^{10} Q(t)\,dt = 20t\Big|_{t=0}^{t=10} + 200\,e^{-t/10}\Big|_{t=0}^{t=10} \approx \frac{200}{e} \approx 67\,\text{kg}.$$