NAME: ____

AM 033 — Applied Mathematics - I

Brown University **Fall 2003**

Midterm Exam #1 **Wednesday, October 7, 2003**

Answer **all** 3 questions on separate paper (provided). Partial credit will be given for correct steps. Begin each solution write-up on a blank sheet of paper. Staple all sheets, including this cover sheet, and hand in. Crib sheets, calculators, and books are not permitted.

[30 pts.] **1.** Solve the ODE

$$(y^2 + 2xy)dx - x^2 dy = 0.$$

[Hint: find an integrating factor.]

Solution:

$$M = y^2 + 2xy \qquad N = -x^2$$

implies

$$M_y = 2y + 2x \qquad N_x = -2x.$$

Thus, $M_y \neq N_x$ and the ODE is not exact. Check for integrating factors $\mu(x)$ or $\mu(y)$:

$$\frac{M_y - N_x}{N} = \frac{2y + 4x}{-x^2} \neq func(x),$$

$$\frac{N_x - M_y}{M} = \frac{-2y - 4x}{y^2 + 2xy} = \frac{-2}{y} = func(y).$$

Thus

$$\frac{d\mu}{dx} = \frac{-2}{y}\mu$$

and

$$\mu = y^{-2}.$$

With this we have

$$\hat{M} = 1 + 2\frac{x}{y} \qquad \hat{N} = -\frac{x^2}{y^2}.$$

Then

$$\begin{aligned}\Psi(x,y) &= \int M dx \\ &= \int 1 + 2\frac{x}{y} dx \\ &= x + \frac{x^2}{y} + h(y).\end{aligned}$$

From $\psi_y = N$ we have

$$-\frac{x^2}{y^2} + h'(y) = -\frac{x^2}{y^2}.$$

Thus $h'(y) = 0$ and $h(y) = \textit{constant}$. Thus $\Psi(x,y) = x + \frac{x^2}{y}$ and $x + \frac{x^2}{y} = c$ defines the solution.

[30 pts.] 2. Consider the ODE

$$\frac{dy}{dx} = \frac{2y}{x} - x^2y^2.$$

Solve using a transformation technique (i.e. recall homogeneous, Bernoulli, and Riccati Equations).

Solution: Bernoulli with $n = 2$. Let $z = y^{1-2} = y^{-1}$. Then $\frac{dz}{dx} = -y^{-2}\frac{dy}{dx}$. Substituting into the ODE we find

$$z' = -\frac{2}{x}z = x^2. \tag{1}$$

This is first order linear in z, so the integrating factor $\mu = e^{\int 2/x} = x^2$. And

$$z = \frac{x^5 + c}{5x^2}. \tag{2}$$

Backsubstituting results in

$$y = \frac{5x^2}{x^5 + c}. \tag{3}$$

[40 pts.] 3. A sailboat has been running (on a straight course) under a light wind at 1 m/sec. Suddenly the wind picks up, blowing hard enough to apply a constant force of 600 N (kg·m/sec^2) to the sailboat. The only other force acting on the boat is water resistance that is proportional to the velocity of the boat. The proportionality constant for water resistance is k=100 kg/sec and the mass of the sailboat is 50 kg.

[10 pts.] (a) Find the equation of motion (IVP) of the sailboat (in terms of the velocity, v). [**Hint:** Recall the equation of motion for a skydiver with viscous dampening is $mv' = mg - kv$.]

[10 pts.] (b) What is the limiting velocity of the sailboat?

[10 pts.] (c) My coffee slides off the dashboard at a velocity of 3.5 m/sec. How how much time do I have from the point at which the wind picks up to catch my coffee mug (in seconds)? [Assume $ln(1/2) \approx 0.7$]

[10 pts.] (d) After the wind picks up, how much distance does the boat travel in 10 seconds. [Assume $e^{-20} \approx 0$].

<u>**Solution:**</u> The IVP is

$$v' = 12 - 2v.v(0) = 1.$$

This is first order linear, so we have

$$\mu = e^{2t}$$
$$v(t) = 6 + c * e^{-2t}.$$

Applying the ICs results in

$$v(t) = 6 - 5e^{-2t}. \tag{4}$$

The limiting velocity is 6 m/sec and the time for a velocity of 3.5 m/sec is approximately $t^* = -2ln(\frac{1}{2})$.
Integrating $v(t)$ to find $x(t)$ yields

$$x(t) = 6t + \frac{5}{2}e^{-2t} + c. \tag{5}$$

The constant is found to be $c = -5/2$ since $x(0) = 0$. From this, $x(10) \approx 60$.