

## AM 033 — Applied Mathematics - I

Brown University  
Homework, Set 9Fall 2003  
Due November 20

9.1 Find the Laplace transforms of the following functions

$$\begin{aligned} \text{(a)} \quad f(t) &= \sin 3t \cos 3t, & \text{(b)} \quad f(t) &= t[H(t-1) - H(t-4)], \\ \text{(c)} \quad f(t) &= 2tH(t-1), & \text{(d)} \quad f(t) &= H(t-2) - H(3-t), \end{aligned}$$

where  $H(t)$  is the Heaviside function:

$$H(t) = \begin{cases} 1, & t > 0 \\ 1/2, & t = 0 \\ 0, & t < 0 \end{cases}$$

**Solution.** (a) Since  $\sin 3t \cos 3t = \frac{1}{2} \sin 6t$ , we obtain

$$f^L(\lambda) = \frac{1}{2} \cdot \frac{6}{\lambda^2 + 36} = \frac{3}{\lambda^2 + 36}.$$

(b) Since

$$H(t-1) - H(t-4) = \begin{cases} 0, & t < 1, \\ 1, & 1 < t < 4, \\ 0, & t > 4, \end{cases}$$

we have

$$f^L(\lambda) = \int_1^4 t e^{-\lambda t} dt = \frac{1}{\lambda} e^{-\lambda} - \frac{4}{\lambda} e^{-4\lambda} - \frac{1}{\lambda^2} e^{-4\lambda} + \frac{1}{\lambda^2} e^{-\lambda}.$$

(c)

$$f^L(\lambda) = \int_1^{\infty} 2t e^{-\lambda t} dt = \frac{2}{\lambda} e^{-\lambda} + \frac{2}{\lambda^2} e^{-\lambda}.$$

(d) Since

$$H(t-2) - H(3-t) = \begin{cases} -1, & 0 < t < 2, \\ 0, & 2 < t < 3, \\ 1, & t > 3, \end{cases}$$

we have

$$f^L(\lambda) = -\int_0^2 e^{-\lambda t} dt + \int_3^{\infty} e^{-\lambda t} dt = \frac{1}{\lambda} e^{-2\lambda} - \frac{1}{\lambda} + \frac{1}{\lambda} e^{-3\lambda}.$$

9.2 Use method of partial fractions to decompose the given rational functions into a sum of simpler expressions.

$$\text{(a)} \quad \frac{2\lambda + 3}{(\lambda - 1)(\lambda - 2)^2}, \quad \text{(b)} \quad \frac{\lambda^2 + 5\lambda - 3}{(\lambda^2 + 16)(\lambda - 2)}.$$

**Solution.**

$$\text{(a)} \quad \frac{2\lambda + 3}{(\lambda - 1)(\lambda - 2)^2} = \frac{5}{\lambda - 1} - \frac{5}{\lambda - 2} + \frac{7}{(\lambda - 2)^2}.$$

$$\text{(b)} \quad \frac{\lambda^2 + 5\lambda - 3}{(\lambda^2 + 16)(\lambda - 2)} = \frac{1}{20} \frac{9\lambda + 118}{\lambda^2 + 16} + \frac{1}{20} \frac{11}{\lambda - 2}.$$

9.3 Find the Laplace transform of the periodic function

$$f(t) = \begin{cases} \sin t, & \text{for } 0 < t < \pi, \\ 0, & \text{for } \pi < t < 2\pi, \\ f(t) = f(t + 2\pi), & \text{for all positive } t. \end{cases}$$

**Solution.** For a periodic function  $f(t)$  with the period  $2\pi$  we use the formula:

$$f^L(\lambda) = \frac{1}{1 - e^{-\lambda 2\pi}} \int_0^{2\pi} f(t) e^{-\lambda t} dt.$$

In our case we have

$$f^L(\lambda) = \frac{1}{1 - e^{-\lambda 2\pi}} \int_0^{\pi} \sin t e^{-\lambda t} dt = \frac{1}{1 - e^{-\lambda 2\pi}} \frac{1}{\lambda^2 + 1} [1 + e^{-\lambda \pi}].$$

9.4 Find the convolution of two functions  $f(t) = t$  and  $g(t) = \sin 2t$ .

**Solution.**

$$\begin{aligned} (t * g)(t) &= \int_0^t (t - \tau) \sin 2\tau d\tau = t \int_0^t \sin 2\tau d\tau - \int_0^t \tau \sin 2\tau d\tau \\ &= -\frac{t}{2} \cos 2\tau \Big|_{\tau=0}^t + \frac{\tau}{2} \cos 2\tau \Big|_{\tau=0}^t - \frac{1}{2} \int_0^t \cos 2\tau d\tau \\ &= -\frac{t}{2} \cos 2t + \frac{t}{2} + \frac{t}{2} \cos 2t - \frac{1}{4} \sin 2t = \frac{t}{2} - \frac{1}{4} \sin 2t. \end{aligned}$$

We can also find the convolution using the Laplace transform:

$$(t * g)^L(\lambda) = \mathcal{L}(t) \cdot \mathcal{L}(\sin 2t) = \frac{1}{\lambda^2} \frac{2}{\lambda^2 + 4} = \frac{1}{2} \frac{1}{\lambda^2} - \frac{1}{4} \frac{2}{\lambda^2 + 4}.$$

Application of the inverse Laplace transform yields

$$(t * g)(t) = \frac{1}{2} \mathcal{L}^{-1} \frac{1}{\lambda^2} - \frac{1}{4} \mathcal{L}^{-1} \frac{2}{\lambda^2 + 4} = \frac{t}{2} - \frac{1}{4} \sin 2t.$$

9.5 Determine the Laplace transform,  $Y(\lambda) = \mathcal{L}[y](\lambda)$ , of the function  $y(t)$  that is a solution of the given initial value problem.

$$\begin{aligned} \text{(a)} \quad & y'' + 4y' + 4y = t, & y(0) = 1, \quad y'(0) = 0; \\ \text{(b)} \quad & y'' - 2y' = H(t - 1), & y(0) = 0, \quad y'(0) = 0. \end{aligned}$$

**Solution.**

$$\text{(a)} \quad Y(\lambda) \equiv y^L(\lambda) = \frac{\lambda + 4}{(\lambda + 2)^2} + \frac{1}{\lambda^2(\lambda + 2)^2}.$$

$$\text{(b)} \quad Y(\lambda) \equiv y^L(\lambda) = \frac{1}{\lambda^2(\lambda - 2)} e^{-\lambda}.$$