AM 033 — Applied Mathematics - I

Brown University **Fall 2003**
Homework, Set 8 **Due November 6**

8.1 Find the solution to

$$y'' + y = H(t)H(\pi - t), \qquad \text{where } H(t) = \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t < 0, \end{cases}$$

subject to the initial conditions $y(0) = y'(0) = 0$ and that y and y' are continuous at $t = \pi$.

Solution. The associated homogeneous equation $y'' + y = 0$ has the general solution: $y_h(t) = C_1 \cos t + C_2 \sin t$, where C_1 and C_2 are arbitrary constants. To find a particular solution of the given non-homogeneous differential equation, we use the Lagrange's method, that is, we are looking for a particular solution in the form:

$$y_p(t) = A(t) \cos t + B(t) \sin t,$$

where $A(t)$ and $B(t)$ are functions that satisfy the following equations:

$$A'(t) \cos t + B'(t) \sin t = 0, \quad -A'(t) \sin t + B'(t) \cos t = f(t), \quad f(t) = \begin{cases} 1 & \text{if } 0 < t < \pi, \\ 0 & \text{otherwise.} \end{cases}$$

Solving for A' and B', we get

$$A'(t) = -f(t) \sin t \quad \Longrightarrow \quad A(t) = -\int_0^t f(t) \sin t \, dt = \begin{cases} \cos t - 1 & \text{if } 0 < t < \pi, \\ -2 & \text{otherwise.} \end{cases}$$

$$B'(t) = f(t) \cos t \quad \Longrightarrow \quad B(t) = \int_0^t f(t) \cos t \, dt = \begin{cases} \sin t & \text{if } 0 < t < \pi, \\ 0 & \text{otherwise.} \end{cases}$$

Substitution into the guessing form, we obtain a particular solution to be

$$y_p(t) = \begin{cases} 1 - \cos t & \text{if } 0 < t < \pi, \\ -2 \cos t & \text{if } \pi \le t. \end{cases}$$

It is easy to check that the function $y_p(t)$ is continuous at $t = \pi$ and it satisfies the initial conditions $y(0) = y'(0) = 0$. Therefore, $y(t) = y_p(t)$ is the solution of the given problem.

8.2 Find the general solution of the differential equation written in factorized form

$$(D - 1)(D + 1)y = 8\, e^{2t}, \qquad D = \frac{d}{dt},$$

applying sequential integration to the first order differential operators $D + 1$ and $D - 1$. That is, solve first the equation $(D - 1)u = 8\, e^{2t}$ and then $(D + 1)y = u$.

Solution. For $u = (D + 1)y$, we have the first order linear differential equation:

$$(D - 1)u = 8\, e^{2t} \qquad \text{or} \qquad \frac{d}{dt}\left[e^{-t}u\right] = 8\, e^{t}$$

because $\mu(t) = e^{-t}$ is an integrating factor. Integration yields

$$u(t) = 8\, e^{2t} + C_1\, e^{t},$$

where C_1 is an arbitrary constant. Then for y we have the first linear order differential equation:

$$(D+1)y = u = 8\, e^{2t} + C_1\, e^{t},$$

which can again be solved with the aid of an intergating factor $\mu(t) = e^{t}$. Therefore we have

$$\frac{d}{dt}\left[e^{t} y\right] = 8\, e^{3t} + C_1\, e^{2t}.$$

We integrate again

$$e^{t} y(t) = \frac{8}{3}\, e^{3t} + \frac{C_1}{2}\, e^{2t} + C_2,$$

or after multiplication by e^{-t} we obtain the general solution to be

$$y(t) = \frac{8}{3}\, e^{2t} + \frac{C_1}{2}\, e^{t} + C_2\, e^{-t}.$$

8.3 Use the method of undetermined coefficients to solve the following differential equations

(a) $(D^2+4)y = 8\,\sin^2 x$, **(b)** $y'' + 2y' - 3y = e^{-x}$,
(c) $y'' + 4y = t^2\,\sin 2t + (6t+7)\,\cos 2t$, **(d)** $(D^2 + 3D + 2)y = 1 + 3x + x^2$.

Solution. (a) The right-hand side function we rewrite as $8\,\sin^2 x = 4 - 4\,\cos 2x$. Using the superposition principle, we break the given differential equation into two equations:

$$(D^2+4)y = 4, \qquad \text{and} \qquad (D^2+4)y = -4\,\cos 2x$$

because functions 4 and $-4\,\cos 2x$ have different control numbers 0 and $2i$, respectively.

Let us consider first the latter equation. The associated homogeneous differential equation $(D^2+4)y = 0$ has the general solution

$$y_h(x) = C_1\,\cos 2x + C_2\,\sin 2x,$$

where C_1 and C_2 are arbitrary constants, since the characteristic equation $\lambda^2 + 4 = 0$ has two complex eigenvalues: $\lambda_{1,2} = \pm 2i$. A particular solution to the non-homogeneous equation $(D^2+4)y = 4$ we seak in the same form as the right-hand side function, namely, as a constant, which is easy to obtain $y_{p1} = 1$.

To find a particular solution to the equation $(D^2+4)y = -4\,\cos 2x$, we observe that the control number $\sigma = 2i$ of the right-hand side function $-4\,\cos 2x$ matches on the of the roots of the characteristic equation. Therefore we look for a particular solution in the form:

$$y_{p2}(x) = x[A\,\cos 2x + B\,\sin 2x],$$

where A and B are some constants to be determined.

Calculations show that $y''_{p2}(x) = 2(-2A\,\sin 2x + 2B\,\cos 2x) - 4y_{p2}$. Substitution $y_{p2}(x)$ and $y''_{p2}(x)$ into the equation, we obtain

$$2(-2A\,\sin 2x + 2B\,\cos 2x) = -4\,\cos 2x.$$

Equating coefficients we get $A = 0$ and $B = -1$. Hence the general solution of the given differential equation is

$$y(x) = 1 - x\,\sin 2x + C_1\,\cos 2x + C_2\,\sin 2x.$$

(b) The characteristic equation $\lambda^2 + 2\lambda - 3 = 0$ for the associated homogeneous equation $y'' + 2y' - 3y = 0$ has two real roots: $\lambda_1 = 1$ and $\lambda_2 = -3$. neither of them matches the control number, $\sigma = -1$, of the right-hand side function e^{-x}. Therefore we are seak a particular solution in the form

$$y_p(x) = A\,e^{-x}.$$

Substitution of y_p into the given equation leads to $A = -1/4$ and the general solution is

$$y(x) = -\frac{1}{4}\,e^{-x} + C_1\,e^{x} + C_2\,e^{-3x}.$$

(c) The control number $\sigma = 2i$ coinsides with the root of the characteristic equation $\lambda^2 + 4 = 0$ for associated homogeneous equation. Therefore we are looking for a particular solution in the form:

$$y_p(t) = t\left[(A_2 + A_1 t + A_0 t^2)\,\cos 2t + (B_2 + B_1 t + B_0 t^2)\,\sin 2t\right],$$

where A_0, A_1, A_2, B_0, B_1, and B_2 are some constants to be determined later. Maple calculations show that

$$\begin{aligned} y''_p(t) &= 2(2A_0 t + A_1)\sin(2t) + 4(A_0 t^2 + A_1 t + A_2)\cos(2t) + 2tA_0\sin(2t) \\ &\quad + 4t(2A_0 t + A_1)\cos(2t) - 4t(A_0 t^2 + A_1 t + A_2)\sin(2t) + 2(2B_0 t + B_1)\cos(2t) \\ &\quad - 4(B_0 t^2 + B_1 t + B_2)\sin(2t) + 2tB_0\cos(2t) - 4t(2B_0 t + B_1)\sin(2t) - 4t(B_0 t^2 + B_1 t + B_2)\cos(2t) \end{aligned}$$

Substitution into the given differential equation yields the LHS

$$\begin{aligned} &6A_0 t\sin(2t) \\ &+2A_1\sin(2t) \\ &+12A_0 t^2\cos(2t) \\ &+8A_1 t\cos(2t) \\ &+4A_2\cos(2t) \\ &+6B_0 t\cos(2t) \\ &+2B_1\cos(2t) \\ &-12B_0 t^2\sin(2t) \\ &-8B_1 t\sin(2t) \\ &-4B_2\sin(2t) \end{aligned}$$

or

$$\begin{aligned}
&(2A_1 - 4B_2)\sin(2t)\\
&+(6A_0 - 8B_1)t\sin(2t)\\
&-12B_0t^2\sin(2t)\\
&+(4A_2 + 2B_1)\cos(2t)\\
&+(8A_1 + 6B_0)t\cos(2t)\\
&+12A_0t^2\cos(2t)
\end{aligned}$$

Equating coeffientts of like powers, we obtain the system of algebraic equations:

$$\begin{aligned}
2A_1 - 4B_2 &= 0\\
6A_0 - 8B_1 &= 0\\
-12B_0 &= 1\\
4A_2 + 2B_1 &= 7\\
8A_1 + 6B_0 &= 6\\
12A_0 &= 0
\end{aligned}$$

We then have $A_0 = 0$, $B_0 = -\frac{1}{12}$, $B_1 = 0$, $A_1 = \frac{13}{16}$, $A_2 = \frac{7}{4}$, and $B_2 = \frac{13}{32}$. We then get the general solution to be

$$y(t) = C_1 \cos 2x + C_2 \sin 2x + t\left[\left(\frac{7}{4} + \frac{13}{16}t\right)\cos 2t + \left(\frac{13}{32} - \frac{1}{12}t^2\right)\sin 2t.\right]$$

(d) We seak a paricular solution of the given differential equation in the form:

$$y_p(x) = a + bx + cx^2.$$

Its derivatives are

$$y_p'(x) = b + 2cx, \quad \text{and} \quad y_p''(x) = 2c.$$

Substitution into the equation yields

$$2c + 3b + 6cx + 2a + 2bx + 2cx^2 = 1 + 3x + x^2.$$

equating coefficients of like powers, we get

$$2c + 2a + 3b = 1,\ 6c + 2b = 3,\ 2c = 1.$$

Therefore $a = b = 0$ and $c = 1/2$ and we obtain the genral solution to be

$$y = \frac{x^2}{2} + C_1\,e^{-x} + C_2\,e^{-2x}.$$

8.4 Verify that the given functions y_1 and y_2 satisfy the corresponding homogeneous equations; then find the general solution.

(a) $x^2\,y'' + xy' + (x^2 - 0.25)y = 3x^{3/2}\,\sin x;\qquad y_1(x) = x^{-1/2}\,\sin x,\quad y_2(x) = x^{-1/2}\,\cos x;$

(b) $(1-x)\,y'' + xy' - y = \sin x,\qquad y_1(x) = e^x,\quad y_2(x) = x.$

Solution. **(a)** To verify that functions y_1 and y_2 satisfy the corresponding homogeneous equation, we need to find first their derivatives:

$$y_1'(x) = x^{-1/2}\,\cos x - \frac{1}{2}\,x^{-3/2}\,\sin x, \qquad y_2'(x) = -x^{-1/2}\,\sin x - \frac{1}{2}\,x^{-3/2}\,\cos x,$$

$$y_1''(x) = \frac{3}{4}\,x^{-5/2}\,\sin x - x^{-3/2}\,\cos x - y_1, \qquad y_2''(x) = \frac{3}{4}\,x^{-5/2}\,\cos x + x^{-3/2}\,\sin x - y_2.$$

Then simple algebra shows that the functions y_1 and y_2 are solutions of the given Bessel differential equation. Moreover, the Wronskian of these functions is $W(y_1, y_2; x) = y_1 y_2' - y_1' y_2 = -x^{-1}$.

We are looking for a particular solution of the non-homogeneous equation in the form:

$$y(x) = A(x)\,y_1(x) + B(x)\,y_2(x),$$

where unknown functions $A(x)$ and $B(x)$ satisfy the following equations:

$$\begin{cases} A'(x)\,y_1 + B'(x)\,y_2 &= 0, \\ A'(x)\,y_1' + B'(x)\,y_2' &= 3x^{-1/2}\,\sin x. \end{cases}$$

Solving, we obtain

$$A'(x) = 3\sin x\cos x, \qquad \Longrightarrow \qquad A(x) = \frac{3}{2}\,\sin^2 x + C_1,$$

$$B'(x) = -3\sin^2 x, \qquad \Longrightarrow \qquad B(x) = \frac{3}{4}\,\sin 2x - \frac{3x}{2} + C_2,$$

where C_1 and C_2 are arbitrary constants. Substitution back leads to the general solution:

$$y(x) = \frac{3}{2}\,x^{-1/2}\,\sin x - \frac{3}{2}\,x^{1/2}\,\cos x + x^{-1/2}\,[C_1\,\sin x + C_2\,\cos x].$$

(b) The Wronskian of these two functions is $W(y_1, y_2; x) = y_1 y_2' - y_1' y_2 = e^x(1-x)$. So we are looking for a particular solution on the interval that does not contain point $x = 1$ in the form:

$$y(x) = A(x)\,y_1(x) + B(x)\,y_2(x),$$

where unknown functions $A(x)$ and $B(x)$ satisfy the following equations:

$$\begin{cases} A'(x)\,y_1 + B'(x)\,y_2 &= 0, \\ A'(x)\,y_1' + B'(x)\,y_2' &= \sin x/(1-x). \end{cases}$$

Solving for A' and B', we obtain

$$A'(x) = -\frac{x\,\sin x}{e^x\,(1-x)^2}, \qquad B'(x) = \frac{\sin x}{(1-x)^2}.$$

Integrating, we get

$$A(x) = -\int \frac{x\,\sin x}{e^x\,(1-x)^2}\,dx + C_1, \qquad B(x) = \int \frac{\sin x}{(1-x)^2}\,dx + C_2,$$

. where C_1 and C_2 are arbitrary constants. So the genral solution is

$$y(x) = -e^x \int \frac{x\,e^{-x}\,\sin x}{(1-x)^2}\,dx + x\int \frac{\sin x}{(1-x)^2}\,dx + C_1\,e^x + C_2\,x.$$