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AM 033 — Applied Mathematics - I

Brown University **Fall 2003**

Homework, Set 7 **Due October 31**

7.1 Write out the characteristic equation for the given differential equation

$$(\mathbf{a})\ \ y'' - 2y' + 6y = 0, \qquad (\mathbf{b})\ \ (D^2 + 5D - 10)y = 0, \quad \text{where } D = d/dx.$$

Solution.

$$(\mathbf{a})\ \ \lambda^2 - 2\lambda + 6 = 0, \qquad (\mathbf{b})\ \ \lambda^2 + 5\lambda - 10 = 0.$$

7.2 The characteristic equation for a certain differential equation is given. Give the form of the general solution.

$$(\mathbf{a})\ \ \lambda^2 + 2\lambda - 3 = 0, \qquad (\mathbf{b})\ \ \lambda^2 - 2\lambda + 2 = 0.$$

Solution. **(a)** The general solution:

$$y(x) = C_1\, e^{x} + C_2\, e^{-3x}.$$

(b) The general solution:

$$y(x) = e^{x}(C_1 \cos(x) + C_2 \sin(x)).$$

7.3 Write the general solution of the differential equation

$$(\mathbf{a})\ \ y'' + y' - 2y = 0, \quad (\mathbf{b})\ \ 25y'' - 20y' + 4y = 0, \quad (\mathbf{c})\ \ y'' + y' + 1.25y = 0.$$

Solution. **(a)**

$$y(t) = C_1\, e^{t} + C_2\, e^{-2t}.$$

(b) The characteristic equation $(5\lambda - 2)^2 = 0$ has one double root, $\lambda = 2/5$, therefore its general solution is

$$y(t) = e^{-2t/5}\, [C_1 + C_2 t]\,.$$

(c)

$$y(t) = e^{-t/2}\, [C_1\, \cos t + C_2\, \sin t]\,, \qquad \text{since } \lambda_{1,2} = -\frac{1}{2} \pm i.$$

7.4 Solve the initial value problems

$$\begin{array}{ll} (\mathbf{a})\ \ y'' + 8\,y' - 9y = 0, & y(1) = 3, \quad y'(1) = -7; \\ (\mathbf{b})\ \ 16\,y'' + 24\,y' + 9y = 0, & y(0) = 1, \quad y'(0) = -11; \\ (\mathbf{c})\ \ 9\,y'' + 9\,y' + 2.5y = 0, & y(0) = 1, \quad y'(0) = -3/2. \end{array}$$

Solution.

(**a**) $y(t) = e^{-9(t-1)} + 2\,e^{t-1}$.

(**b**) The characteristic equation is $(4\lambda + 3)^2 = 0$. Therefore the general solution is

$$y(t) = (C_1 + C_2 t)\, e^{-3t/4} \qquad \Longrightarrow \qquad y(t) = \left(1 - t\, 10\frac{1}{4}\right)\, e^{-3t/4}.$$

(**c**) $y(t) = e^{-t/2}\left[\sin\dfrac{t}{6} - 6\,\cos\dfrac{t}{6}\right]$.

7.5 Find a second solution of the given Bessel equation

$$x^2\, y'' + x\, y' + (x^2 - 0.25)y = 0, \qquad x > 0, \qquad y_1(x) = x^{-1/2}\,\sin x.$$

Solution. We change the dependent variable by setting $y(x) = v(x)y_1(x)$, where $v(x)$ should be determined. Substition of y and its derivatives into the given differential equation yields

$$x^2\,[v'' y_1 + 2v' y_1' + v y_1''] + x\,[v' y_1 + v y_1'] + v(x)(x^2 - 0.25)y_1 = 0.$$

We regroup terms containg v, v', and v'' to obtain

$$x^2\, y_1\, v'' + v'[2x^2\, y_1' + x y_1] + v[x^2\, y_1'' + x\, y_1' + (x^2 - 0.25)y_1] = 0.$$

Since y_1 is a solution of the given differential equation, we have

$$x^2\, y_1\, u' + u[2x^2\, y_1' + x y_1] = 0, \qquad \text{where } u = v'.$$

Separation of variables yields

$$\frac{du}{u} = -\left(\frac{y_1'}{y_1} + \frac{1}{x}\right)\, dx, \qquad \Longrightarrow \qquad \ln u = -2\ln y_1 - \ln x = -\ln(x y_1^2).$$

In the above integration, we ignored arbitrary constant because we are looking for just another solution. Hence

$$v' = u = \frac{1}{x y_1^2} = \frac{1}{\sin^2 x}.$$

One more integration leads to

$$v(x) = \int \frac{1}{\sin^2 x} = -\cot x,$$

and another linearly independent solution will be

$$y_2(x) = v(x) y_1(x) = -x^{-1/2}\,\cos x.$$