AM 033 — Applied Mathematics - I

Brown University **Fall 2003**

Homework, Set 6 **SOLUTIONS** **Due October 23**

6.1 Classify the differential equation as being either linear or nonlinear. Furthermore, classify the linear ones as being homogeneous or nonhomogeneous, with constant coefficients or with variable coefficients, and state its order.

$$(\mathbf{a})\ \ y'' + x^2 y = 0, \qquad (\mathbf{b})\ \ y''' + xy = \sin x,$$
$$(\mathbf{c})\ \ y \cdot y' = 1, \qquad (\mathbf{d})\ \ y^{(5)} - y^{(4)} + y' = 2x^2 + 3$$

Solution. (a) Linear, homogeneous, variable coefficients, of second order.
(b) Linear, non-homogeneous, variable coefficients, of third order.
(c) Nonlinear of first order.
(d) Linear, non-homogeneous, constant coefficients, of fifth order.

6.2 For each of the following differential equations state the order of the equation.

$$(\mathbf{a})\ \ y''' + xy'' - y^2 = \sin x, \qquad (\mathbf{b})\ \ \sin(y'') + yy^{(4)} = 1.$$

Solution. (a) 3; **(b)** 4.

6.3 Using the symbol $D = \frac{d}{dt}$, rewrite the given differential equation $y'' + 4y' + y = 0$ in the operator form.

Solution.

$$(D^2 + 4D + 1)y = 0.$$

6.4 In each of the following Cauchy problems, determine, without solving the problem, an interval in which the solution is certain to exist.

$$(\mathbf{a})\ \ (x-3)y'' + (\ln x)y = x^2, \quad y(1) = y'(1) = 2, \qquad (\mathbf{b})\ \ (x^2+1)y'' + (x-1)y' + y = 0, \quad y(0) = y'(0) = 1.$$

Solution. (a) $0 < x < 3$. **(b)** Everywhere.

6.5 Evaluate (the symbol D stands for $\frac{d}{dx}$)

$$(\mathbf{a})\ \ (D-2)(x^3 + 2x), \qquad (\mathbf{b})\ \ (D-x)(x^2 - 2x).$$

Solution.

(a) $(D-2)(x^3 + 2x) = D(x^3 + 2x) - 2(x^3 + 2x) = 3x^2 + 2 - 2x^3 - 4x = -2x^3 + 3x^2 - 4x + 2.$
(b) $(D-x)(x^2 - 2x) = D(x^2 - 2x) - x(x^2 - 2x) = 2x - 2 - x^3 + 2x^2 = -x^3 + 2x^2 + 2x - 2.$

6.6 Determine whether the given pair of functions is linearly independent or linearly dependent (on real line)

$$f_1(x) = e^x,\ f_2(x) = e^{2x}.$$

Solution. Linearly independent.

6.7 Obtain the Wronskian of the following three functions

$$f_1(x) = e^x, \quad f_2(x) = x\, e^x, \quad \text{and} \quad f_3(x) = (2x-1)\, e^x.$$

Solution. The Wronskian is

$$W(x) = \det \begin{bmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1'(x) & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{bmatrix} = \begin{vmatrix} e^x & x\,e^x & (2x-1)\,e^x \\ e^x & (1+x)\,e^x & (2x+1)\,e^x \\ e^x & (2+x)\,e^x & (2x+3)\,e^x \end{vmatrix} = e^{3x} \begin{vmatrix} 1 & x & 2x-1 \\ 1 & 1+x & 2x+1 \\ 1 & 2+x & 2x+3 \end{vmatrix} = 0.$$

The result follows immediately from $f_3(x) = 2f_2(x) - f_1(x)$.

6.8 Find the Wronskian of two solutions of the given differential equation without solving the equation.

$$x^2\, y'' + x\, y' + (x^2 - \nu^2) y = 0.$$

Solution. We rewrite the given differential equation in the following form:

$$y'' + p(x)\, y' + q(x)\, y = 0, \qquad p(x) = x^{-1},\ q(x) = 1 - \frac{\nu^2}{x^2}.$$

According to Abel's theorem, the Wronskian is equal to

$$W(x) = C\, \exp\left[- \int p(x)\, dx \right] = C\, \exp\left[-\ln x \right] = \frac{C}{x},$$

where C is some constant.

6.9 The equation $a(x)\, y'' + b(x)\, y' + c(x)\, y = 0$ is said to be **exact** if $a''(x) - b'(x) + c(x) = 0$. Then the exact equation can be rewriten in the form $[a(x)\, y']' + [f(x)\, y(x)]' = 0$ for some function $f(x)$. Determine whether the given differential equation is exact.

$$x^2\, y'' + x\, y' - y = 0.$$

Solution. Exact since it can be rewritten as

$$\frac{d}{dx}\, \left[x^2\, y' - xy \right] = 0.$$