AM 033 — Applied Mathematics - I

Brown University **Fall 2003**

Homework, Set 3 **SOLUTIONS** **Due September 25**

3.1 Any body moving through a fluid such as water or air creates a drag force that tends to retard its motion. Calculations predict and experiments confirm that in air, the drag force on a parachutist/skydiver or a mass fired into earth can be well approximated by a force, called *Newtonian damping*, that is proportional to the square of the magnitude of its velocity v^2, namely, $F = -k\,v^2$, with the Newtonian damping constant k. Balancing the forces of acceleration, gravity, and air resistance leads to the nonlinear differential equation of motion

$$m\,\frac{dv}{dt} = -mg + k\,v^2, \quad \text{or} \quad \frac{dv}{dt} = -g + \frac{k}{m}\,v^2$$

with the homogeneous initial condition $v(0) = 0$. **Note:** Maple may be quite useful in assisting you with this problem.

If the mass of the equipped parachutist is 100 kg, the the Newtonian damping constant k is about 0.2 kg/m. Typical questions to be addressed in the analysis of the "parachute problem" include the following:

(a) Training jumps for the Parachute team begin 1,200 meters above ground level (AGL). What is the terminal velocity ($\lim_{t\to\infty} v(t)$)?

(b) The free-fall portion of the jump lasts about 10 seconds; free-falls longer than 13 seconds are grounds for removal from the team. What is the velocity of the skydiver after 10 seconds of free-fall? What is his/her altitude AGL?

(c) If a malfunction occurs with the main chute, almost 6 seconds will be required to recognize and react to the problem and to deploy the reserve parachute. What is the velocity and the altitude of the skydiver after 16 seconds of free-fall?

(d) The parachute requires approximately 3.2 seconds to fully deploy from the time the ripcord is pulled. For simplicity, we ignore this time and assume that the chute is deployed immediately. The Newtonian damping constant is not actualy a constant but is of the following form:

$$k(t) = \begin{cases} k_1, & \text{if } t < t_d, \\ 50k_1, & \text{if } t > t_d, \end{cases}$$

where $k_1 = 0.2$. What is the velocity at impact if the parachute was deployed after 10 seconds?

(e) The landing velocity should be no worse than a free-fall from 1.5 meters wall–between 4.6 and 5.2 m/sec. What is the latest time that the parachute can be open while keeping the impact velocity below a specified threshold?

Solution.

3.2 Solve the initial value problems

(a) $(9x^2 + y - 1)dx = (4y - x)dy$, $y(1) = 0$; (b) $y' = e^x + 2y$, $y(0) = -1$;

(c) $3x^{-1}y^2\,dx - 2y\,dy = 0, \quad y(4) = 8$; (d) $(2x\,e^x - y^2)\,dx + 2y\,dy = 0, \quad y(0) = \sqrt{2}$.

Solution. **(a)** The given equation is exact, so we integrate from the point $(1,0)$ to (x,y) along horizontal line and then along the vertical line. This yields

$$\psi(x,y) \equiv \int_{(1,0)}^{(x,y)} (9x^2+y-1)dx + (x-4y)\,dy = \int_1^x (9x^2-1)\,dx + \int_0^y (x-4y)\,dy.$$

Equating the potential function, $\psi(x,y)$ to zero, we obtain the solution of the given Cauchy problem:

$$\psi(x,y) = (3x^3-x)\Big|_{x=1}^{x} - (2y^2-xy)\Big|_{y=0}^{y} = 0 \quad \text{or} \quad 3x^3 - x - 2y^2 + xy = 2.$$

(b) We rewrite the differential equation in differentials: $(e^x+2y)\,dx - dy = 0$.So there exists an integrating factor as a function of x only since

$$(M_x - N_y)/N = -2, \qquad \text{where } M(x,y) = e^x + 2y, \quad N(x,y) = -1.$$

Myltiplying both sides by $\mu(x) = e^{-2x}$, we reduce the equation to athe exact one:

$$(e^{-x} + 2y\,e^{-2x})\,dx - e^{-2x}\,dy = 0.$$

Its potential function is $\psi(x,y) = -e^{-x} - y\,e^{-2x}$ and therefore the general solution is $e^{-x} + y\,e^{-2x} = C$ for some arbitrary constant C. From the initial condition follows that $C=0$ and therefore

$$y(x) = -e^x.$$

(c) The given differential equation is of the form $M\,dx + N\,dy = 0$ where $M(x,y) = 3x^{-1}y^2$ and $N(x,y) = -2y$. Since the ration

$$\frac{M_x - N_y}{N} = \frac{6x^{-1}y}{-2y} = -\frac{3}{x}$$

is a function of x only, an integrating factor is $\mu(x) = x^{-3}$. After multiplication by $\mu(x)$, we obtain the exact equation:

$$3\,\frac{y^2}{x^4}\,dx - \frac{2y}{x^3}\,dy = 0.$$

So the general solution is

$$-\frac{y^2}{x^3} = C \qquad \text{or} \qquad y^2 = C\,x^3.$$

From the initial condition follows that $C = 1$.

(d) Multiplying both side of the equation by the integrating factor $\mu(x) = e^{-x}$ we reduce the equation to the exact one:

$$(2x - y^2\,e^{-x})\,dx + 2y\,e^{-x}\,dy = 0.$$

Its potential function is $\psi(x,y) = x^2 - y^2\,e^{-x}$, so the solution of the given initial value problem is

$$x^2 - y^2\,e^{-x} = -2.$$