CS273: Theory of Computation, Summer 2008

Some known decidable languages:

$$
\begin{aligned}
A_{\text{DFA}} &= \{\langle M, w\rangle \mid M \text{ is a DFA and } w \in L(M)\}\\
E_{\text{DFA}} &= \{\langle M\rangle \mid M \text{ is a DFA and } L(M) = \emptyset\}\\
EQ_{\text{DFA}} &= \{\langle M_1, M_2\rangle \mid M_1, M_2 \text{ are DFAs and } L(M_1) = L(M_2)\}\\
A_{\text{CFG}} &= \{\langle G, w\rangle \mid G \text{ is a CFG and } w \in L(G)\}\\
E_{\text{CFG}} &= \{\langle G\rangle \mid G \text{ is a CFG and } L(G) = \emptyset\}
\end{aligned}
$$

Some known undecidable languages:

$$
\begin{aligned}
A_{\text{TM}} &= \{\langle M, w\rangle \mid M \text{ is a TM and } w \in L(M)\}\\
E_{\text{TM}} &= \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \emptyset\}\\
ALL_{\text{CFG}} &= \{\langle G\rangle \mid G \text{ is a CFG and } L(G) = \Sigma^*\}\\
EQ_{\text{CFG}} &= \{\langle G_1, G_2\rangle \mid G_1, G_2 \text{ are CFGs and } L(G_1) = L(G_2)\}
\end{aligned}
$$

Suggestion: Alternate between the parts of problems 1 & 2, to practice both kinds of reductions.

1. Use reductions to show that the following languages are **decidable**:

   (a) $\{\langle M\rangle \mid M \text{ is a DFA and } L(M) = \Sigma^*\}$
   (b) $\{\langle M\rangle \mid M \text{ is a DFA and } L(M) = L(a^*b^*)\}$
   (c) $\{\langle M_1, M_2, G\rangle \mid M_1, M_2 \text{ are DFAs, } G \text{ is a CFG, and } M_1 \text{ accepts some string in } L(M_2) \circ L(G)\}$
   (d) $\{\langle M_1, M_2, M_3\rangle \mid M_1, M_2, M_3 \text{ are DFAs and } L(M_1) \subseteq L(M_2) \subseteq L(M_3)\}$

   Recall: To show that A is decidable, give a reduction from A to a known *decidable* language B. That is, describe a decider for A that uses the decider for B as a subroutine.

2. Use reductions[1] to show that the following languages are **undecidable**:

   (a) $\{\langle M, w\rangle \mid M \text{ is a TM, } w \text{ is a palindrome, and } L(M) = \emptyset\}$
   (b) $\{\langle M, n\rangle \mid M \text{ is a TM and } a^n b^n \in L(M)\}$
   (c) $\{\langle G_1, G_2, G_3\rangle \mid G_1, G_2, G_3 \text{ are CFGs and } L(G_1) \cap L(G_2) = L(G_3)\}$
   (d) $\{\langle G_1, G_2, G_3\rangle \mid G_1, G_2, G_3 \text{ are CFGs and } L(G_1) \subseteq L(G_2) \subseteq L(G_3)\}$

   Recall: To show that A is undecidable, give a reduction from a known *undecidable* language B to A. That is, describe a decider for B that uses a (hypothetical) decider for A as a subroutine.

   Hint: In your reductions on parts (c) and (d), run the hypothetical decider on some choice of $\langle G_1, G_2, G_3\rangle$ where two of these three CFGs are the same CFG.

3. In this problem, let $M_1, M_2, \ldots$ be an enumeration (list) of all possible Turing Machines.

   (a) Prove that $\{a^n b^n \mid a^n b^n \notin L(M_n)\}$ is not Turing-recognizable.
   (b) Prove that there is an unrecognizable subset of $\{a^n b^{n^2} c^{2^n} \mid n \geq 0\}$. *Hint:* modify the argument from part (a).
   (c) Challenge: Prove that every infinite language has an unrecognizable subset.

[1] By Rice's theorem, we know that the language in part (b) is undecidable (it is a nontrivial property of TM languages). For practice, though, try to prove it directly via a reduction.