Theorem (termination of simply-typed λ-calculus expressions):
Define the predicate $\Gamma \vdash e \Downarrow^*$ which means that e when applied to any full sequence of inputs terminates. For example, if $\Gamma \vdash e : \bullet \to \bullet \to \bullet$, then we require that for all expressions a and b of type $\bullet$, $e\ a\ b$ terminates. More formally,

$$\begin{array}{lll} (\Gamma \vdash e \Downarrow^*) \ \Leftrightarrow & (\Gamma \vdash e : \bullet) & \Rightarrow \ (\exists v \ . \ e \Downarrow v) \\ & \wedge(\Gamma \vdash e : \alpha \to \beta) & \Rightarrow \ (\forall a \ . \ (\Gamma \vdash a : \alpha) \wedge (\Gamma \vdash a \Downarrow^*) \ \Rightarrow \ (\Gamma \vdash e\ a \Downarrow^*)) \end{array}$$

Note that the fact that we eventually "run out" of arguments for e is critical, as this ensures that the we can only expand the subexpression $(\Gamma \vdash e\ a \Downarrow^*)$ a bounded number of times.

Now consider any expression e; it may have one of three forms:

a free variable,	x
an application,	$e_1\ e_2$
a λ-abstraction,	$\lambda x : \alpha . e'$

Case 1: $e = x$. If e is a free variable, then by definition, $\Gamma \vdash e \Downarrow^*$. This is because we cannot perform β-reduction on expressions of the form $x\ a\ b \ldots$, hence provided $a, b, \ldots$ terminate, so does e.

Case 2: $e = e_1\ e_2$. If e is an application, $e_1\ e_2$, we can assume inductively that $\Gamma \vdash e_1 \Downarrow^*$ and $\Gamma \vdash e_2 \Downarrow^*$, where $\Gamma \vdash e_1 : \alpha \to \beta$ and $\Gamma \vdash e_2 : \alpha$. By definition of $\Downarrow^*$, we have that

$$(\Gamma \vdash e_1 \Downarrow^*) \ \Leftrightarrow \ ((\Gamma \vdash e_2 \Downarrow^*) \ \Rightarrow \ (\Gamma \vdash e_1\ e_2 \Downarrow^*))$$

Therefore, $\Gamma \vdash e_1\ e_2 \Downarrow^*$, *i.e.,* $\Gamma \vdash e \Downarrow^*$.

Case 3: $e = \lambda x : \alpha . e'$. Next assume $e = \lambda x : \alpha . e'$. Observe that $\Gamma \vdash e : \alpha \to \beta$ for some type β, thus $\Gamma, x : \alpha \vdash e' : \beta$. By induction, we can assume that $\Gamma, x : \alpha \vdash e' \Downarrow^*$. We will derive a lemma that we can use together with this inductive assumption to prove that $\Gamma \vdash e \Downarrow^*$.
Note that by the definition of $\Downarrow^*$,

$$(\Gamma \vdash e \Downarrow^*) \ \Leftrightarrow \ (\forall a \ . \ (\Gamma \vdash a : \alpha) \wedge (\Gamma \vdash a \Downarrow^*) \ \Rightarrow \ (\Gamma \vdash e\ a \Downarrow^*))$$

Taking one step of β reduction,

$$(\Gamma \vdash e \Downarrow^*) \ \Leftrightarrow \ (\forall a \ . \ (\Gamma \vdash a : \alpha) \wedge (\Gamma \vdash a \Downarrow^*) \ \Rightarrow \ (\Gamma \vdash [x \mapsto a]e' \Downarrow^*))$$

Lemma (termination extensionality):
The well-typed expression $[x \mapsto a]e'$ terminates if and only if e' itself terminates, where x is a free variable of type α.

$$(\forall a \ . \ (\Gamma \vdash a : \alpha) \wedge (\Gamma \vdash a \Downarrow^*) \ \Rightarrow \ (\Gamma \vdash [x \mapsto a]e' \Downarrow^*)) \ \Leftrightarrow \ (\Gamma, x : \alpha \vdash e' \Downarrow^*)$$

We can prove this by a second induction, on the structure of e'.

Case 1L: $e' = x$ (base case). In this case, the lemma statement simplifies to the following tautology (both sides of the $\Leftrightarrow$ are true):

$$(\forall a\ .\ (\Gamma \vdash a : \alpha) \land (\Gamma \vdash a \Downarrow^*) \ \Rightarrow\ (\Gamma \vdash a \Downarrow^*)) \ \Leftrightarrow\ (\Gamma, x : \alpha \vdash x \Downarrow^*)$$

Case 2L: $e' = y \neq x$ (base case). In this case, the lemma statement simplifies to the following tautology:

$$(\forall a\ .\ (\Gamma \vdash a : \alpha) \land (\Gamma \vdash a \Downarrow^*) \ \Rightarrow\ (\Gamma \vdash y \Downarrow^*)) \ \Leftrightarrow\ (\Gamma, x : \alpha \vdash y \Downarrow^*)$$

Case 3L: $e' = \lambda x.e''$. In this case, the lemma statement simplifies to the following tautology:

$$(\forall a\ .\ (\Gamma \vdash a : \alpha) \land (\Gamma \vdash a \Downarrow^*) \ \Rightarrow\ (\Gamma \vdash \lambda x.e'' \Downarrow^*)) \ \Leftrightarrow\ (\Gamma, x : \alpha \vdash \lambda x.e'' \Downarrow^*)$$

Case 4L: $e' = \lambda y : \gamma.e''$ where $x \neq y$. In this case, the lemma statement simplifies to the following:

$$(\forall a\ .\ (\Gamma \vdash a : \alpha) \land (\Gamma \vdash a \Downarrow^*) \ \Rightarrow\ (\Gamma \vdash \lambda y : \gamma.[x \mapsto a]e'' \Downarrow^*)) \ \Leftrightarrow\ (\Gamma, x : \alpha \vdash \lambda y : \gamma.e'' \Downarrow^*)$$

By induction, we can assume the following:

$$\begin{aligned}
(\Gamma \vdash \lambda y : \gamma.[x \mapsto a]e'' \Downarrow^*) \ &\Leftrightarrow\ (\Gamma, y : \gamma \vdash [x \mapsto a]e'' \Downarrow^*) \\
(\Gamma, x : \alpha \vdash \lambda y : \gamma.e'' \Downarrow^*) \ &\Leftrightarrow\ (\Gamma, x : \alpha, y : \gamma \vdash e'' \Downarrow^*)
\end{aligned}$$

Thus we can further simplify the lemma statement to:

$$(\forall a\ .\ (\Gamma \vdash a : \alpha) \land (\Gamma \vdash a \Downarrow^*) \ \Rightarrow\ (\Gamma, y : \gamma \vdash [x \mapsto a]e'' \Downarrow^*)) \ \Leftrightarrow\ (\Gamma, x : \alpha, y : \gamma \vdash e'' \Downarrow^*)$$

Now, via a second appeal to induction, the above is proven.

Case 5L: $e' = e_1\ e_2$. This time, the lemma can be stated as follows:

$$(\forall a\ .\ (\Gamma \vdash a : \alpha) \land (\Gamma \vdash a \Downarrow^*) \ \Rightarrow\ (\Gamma \vdash [x \mapsto a]e_1\ e_2 \Downarrow^*)) \ \Leftrightarrow\ (\Gamma, x : \alpha \vdash e_1\ e_2 \Downarrow^*)$$

By the definitions of substitution and $\Downarrow^*$, the above can be restated as follows:

$$\begin{aligned}
&(\forall a\ .\ (\Gamma \vdash a : \alpha) \land (\Gamma \vdash a \Downarrow^*) \ \Rightarrow\ (\Gamma \vdash [x \mapsto a]e_1 \Downarrow^*) \land (\Gamma \vdash [x \mapsto a]e_2 \Downarrow^*)) \\
&\quad \Leftrightarrow\ (\Gamma, x : \alpha \vdash e_1 \Downarrow^*) \land (\Gamma, x : \alpha \vdash e_2 \Downarrow^*)
\end{aligned}$$

The above is entailed by the following, more general statement:

$$\begin{aligned}
&(\forall a\ .\ (\Gamma \vdash a : \alpha) \land (\Gamma \vdash a \Downarrow^*) \ \Rightarrow\ (\Gamma \vdash [x \mapsto a]e_1 \Downarrow^*)) \ \Leftrightarrow\ (\Gamma, x : \alpha \vdash e_1 \Downarrow^*) \\
\land&(\forall a\ .\ (\Gamma \vdash a : \alpha) \land (\Gamma \vdash a \Downarrow^*) \ \Rightarrow\ (\Gamma \vdash [x \mapsto a]e_2 \Downarrow^*)) \ \Leftrightarrow\ (\Gamma, x : \alpha \vdash e_2 \Downarrow^*)
\end{aligned}$$

Note that each of the two conjuncts above holds by induction.
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