Barycentric Lagrange Interpolation

J.P. Berrut and L.N. Trefethen
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Stephen Bond

UIUC

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Monomial Basis Interpolation: Example

- Interpolate 3 Data Points

$$(t_1, y_1) \qquad (t_2, y_2) \qquad (t_3, y_3)$$

- Monomial Basis

$$p(t) = x_1 + x_2 t + x_3 t^2$$

- Cost to determine coefficients?

$$\begin{bmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

- Cost to evaluate polynomial?

$$p(t) = x_1 + t(x_2 + x_3 t) \qquad \text{Horner's Method}$$

Lagrange Basis Interpolation: Example

- Interpolate 3 Data Points

$$(t_1, y_1) \qquad (t_2, y_2) \qquad (t_3, y_3)$$

- Lagrange Basis

$$p(t) = y_1 \frac{(t - t_2)(t - t_3)}{(t_1 - t_2)(t_1 - t_3)} + y_2 \frac{(t - t_1)(t - t_3)}{(t_2 - t_1)(t_2 - t_3)} + y_3 \frac{(t - t_1)(t - t_2)}{(t_3 - t_1)(t_3 - t_2)}$$

- Cost to determine coefficients?

- Cost to evaluate polynomial?

- Interpolate 3 Data Points

$$(t_1, y_1) \qquad (t_2, y_2) \qquad (t_3, y_3)$$

- Barycentric Lagrange Basis

$$\begin{aligned} p(t) = {} & y_1 \frac{(t - t_1)(t - t_2)(t - t_3)}{(t_1 - t_2)(t_1 - t_3)(t - t_1)} \\ & + y_2 \frac{(t - t_1)(t - t_2)(t - t_3)}{(t_2 - t_1)(t_2 - t_3)(t - t_2)} \\ & + y_3 \frac{(t - t_1)(t - t_2)(t - t_3)}{(t_3 - t_1)(t_3 - t_2)(t - t_3)} \end{aligned}$$

- Commmon term

$$\ell(t) = (t - t_1)(t - t_2)(t - t_3)$$

can be factored out!

- Lagrange Basis

$$p(t) = y_1(t-t_2)(t-t_3)w_1 + y_2(t-t_1)(t-t_3)w_2 + y_3(t-t_1)(t-t_2)w_3$$

- Weight Constants

$$w_1 = \frac{1}{(t_1 - t_2)(t_1 - t_3)}, \quad w_2 = \frac{1}{(t_2 - t_1)(t_2 - t_3)},$$

$$w_3 = \frac{1}{(t_3 - t_1)(t_3 - t_2)}$$

- Barycentric Lagrange Basis

$$p(t) = \ell(t)\left(y_1 \frac{w_1}{t - t_1} + y_2 \frac{w_2}{t - t_2} + y_3 \frac{w_3}{t - t_3}\right)$$

$$\ell(t) = (t - t_1)(t - t_2)(t - t_3)$$

- Cost to determine weight constants, w_i:

$$(n \text{ weights}) \times (2n - 2 \text{ operations per weight}) = \mathcal{O}[n^2] \text{ operations},$$

which is the same complexity as the Newton basis.

- Cost to evaluate polynomial for given weight constants:

$$\ell(t) \text{ costs } \mathcal{O}[n] \qquad \text{and} \qquad \sum_{i=1}^{n} y_i \frac{w_i}{t - t_i} \text{ costs } \mathcal{O}[n],$$

which is the same complexity as the Newton basis.

- Cost to update when adding an additional point:

$$\mathcal{O}[n] \text{ for } w_1, \ldots, w_n \qquad \text{and} \qquad \mathcal{O}[n] \text{ for } w_{n+1},$$

which is the same complexity as the Newton basis.

Barycentric Lagrange: Stability

- What happens when $t = t_i$?

  Use an "if statement" and return y_i to avoid $0/0$.

- What happens as $t \longrightarrow t_i$?

  Careful analysis shows this is not a problem in floating-point.

  (See Henrici 1979 and Higham 2004)

- Barycentric Lagrange is not "point-order" dependent.

  Same representation independent of point ordering.

  This is in contrast to the Newton basis.

- Weights depend only on t_i's not on y_i's.

  Can be precomputed if interpolation points are fixed.

Consider "Barycentric Lagrange" for your interpolation problem.