Released: January 29, 2008
Due: February 7, 2008

Problem 1:

We say that a complexity class $\mathbf{X}$ is *closed downward under Karp reductions* if:

$$\text{for all languages } A, B\text{: } A \in \mathbf{X} \text{ and } B \leq_P A \implies B \in \mathbf{X}$$

Show that $\mathbf{E}$ and $\mathbf{NE}$ are *not* closed downward under Karp reductions. (These two complexity classes were defined in the previous problem set.)

Problem 2:

Show that $\mathbf{E} = \mathbf{NE}$ implies $\mathbf{EXP} = \mathbf{NEXP}$.

Problem 3:

Show that $\overline{\mathsf{SAT}}$ (the complement of SAT) is $\mathbf{NP}$-hard under Cook reductions. That is, every language in $\mathbf{NP}$ reduces to $\overline{\mathsf{SAT}}$ via a Cook reduction.

Problem 4:

Show that $\mathbf{P} \neq \mathbf{DSPACE}(n)$.

Hint: Show that one class is closed downward under Karp reductions, while the other is not.

Problem 5 (AB chapter 2, #11b):

Give a parsimonious Karp reduction from SAT to $\mathsf{3SAT}$.

Problem 6:

In this problem, we analyze a reduction from $\mathsf{3SAT}$ to the following language:

$$\mathsf{MAX\text{-}2SAT} = \{(\phi, k) \mid \phi \text{ is a 2-CNF formula, and there is an assignment that satisfies at least } k \text{ clauses}\}$$

Our reduction is the following: Given a $\mathsf{3SAT}$ instance ϕ, we will output a $\mathsf{MAX\text{-}2SAT}$ instance (ϕ', k), where ϕ' is a 2-CNF formula. To construct ϕ', do the following: for each clause $(x \vee y \vee z)$ in ϕ, add the following 10 clauses to ϕ' (where w is a fresh variable for each clause):

$$(x), (y), (z), (\neg x \vee \neg y), (\neg y \vee \neg z), (\neg x \vee \neg z), (w), (x \vee, \neg w), (y \vee \neg w), (z \vee \neg w)$$

Find a value of k such that $(\phi', k) \in \mathsf{MAX\text{-}2SAT}$ if and only if $\phi \in \mathsf{3SAT}$. Prove the correctness of the reduction.

Problem 7 (Extra credit):

Consider the following language:

$$\mathsf{MAX\text{-}CUT} = \{(G, k) \mid G \text{ is a multigraph with a cut of size at least } k\}$$

A *cut* in a graph is a partition of its vertices into two parts. The size of the cut is the number of edges which "cross" the cut (whose endpoints are in opposite parts). A multigraph means we allow duplicate edges.

We now analyze a reduction from $\mathsf{MAX\text{-}2SAT}$ to $\mathsf{MAX\text{-}CUT}$. Given an instance (ϕ, k) of $\mathsf{MAX\text{-}2SAT}$, let n be the number of variables occuring in ϕ, and m the number of clauses. Consider the following graph:

G_ϕ is a graph with a vertices labeled x_i and $\neg x_i$ for each variable x occuring in ϕ, and two special vertices labeled T and F. We add $5m$ edges between T and F, and $5m$ edges between each pair $(x_i, \neg x_i)$ — see Figure 1. Then, for each clause $(x \vee y) \in \phi$, where x and y are literals, we add the following 7 edges (see Figure 2):

- $(x, y), (T, x), (T, y)$.

(a) Show that in the largest cut in G_ϕ, T and F must be in opposite parts.

(b) Show that in the largest cut in G_ϕ, the vertices corresponding to x and $\neg x$ must be in opposite parts.

(c) Argue that $(\phi, k) \in \mathsf{MAX\text{-}2SAT}$ if and only if $(G_\phi, 5m + 5mn + 4k + 2(m - k)) \in \mathsf{MAX\text{-}CUT}$.

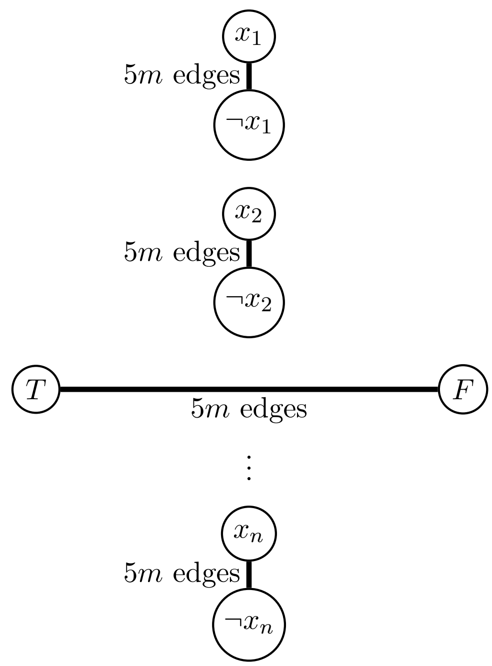

Figure 1: Starting graph for G_ϕ, where ϕ has n variables, $x_1, \ldots, x_n$.

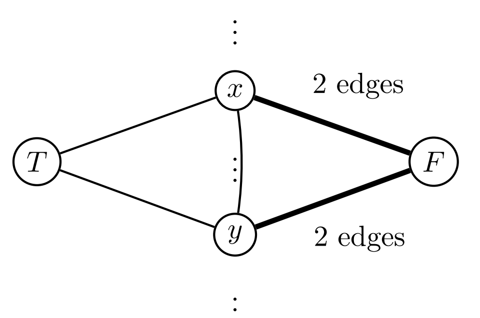

Figure 2: Edges to add for a clause of the form $(x \vee y)$