

CS 273: Intro to Theory of Computation, Fall 2007

Quiz 2 Solutions

1. (1 points) A PDA is basically an NFA with one extra feature. What's the extra feature?

Solution: a stack.

2. (2 points) Is the language $\{a^n b^m \mid n \in \mathbb{N}, m \in \mathbb{N}\}$ regular? Why or why not?

Solution: Yes, because the number of a's doesn't have to match the number of b's, so this language is just $a^+ b^+$.

It's a mistake to argue that it must be regular because it satisfies the pumping lemma. The pumping lemma only works one direction: if it's regular, it satisfies the pumping lemma. There are some non-regular languages that also satisfy the pumping lemma.

Some people thought it was $a^* b^*$, because they forget that we are defining \mathbb{N} not to contain zero. (I didn't take off points for this.)

3. (4 points) Give a context-free grammar that generates each of the following subsets of $\{a, b, c\}^*$. Just give the rule(s) for each grammar. Assume that the start symbol is always S and variables are uppercase letters.

(a) $\{a^n b^n \mid n \geq 0\}$

Solution: $S \rightarrow aSb \mid \epsilon$

(b) $c^* \cup b^*$

Solution:

$$S \rightarrow B \mid C$$

$$B \rightarrow bB \mid \epsilon$$

$$C \rightarrow cC \mid \epsilon$$

Many people had grammars that generated closely-related languages. An especially common mistake was to generate $(c \cup b)^*$ in part (b).

4. (2 points) What language is generated by the following context-free grammar, with start symbol S ?

$$S \rightarrow AA$$

$$A \rightarrow a \mid b$$

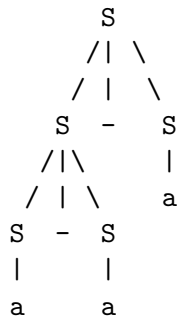
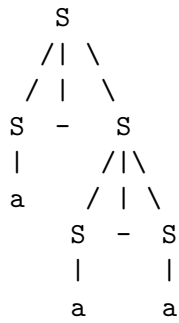
Solution: $\{aa, ab, ba, bb\}$

5. (4 points) Give an example of a context-free grammar G which is ambiguous. Explain why it is ambiguous by showing two parse trees yielding the same string.

Solution: There are quite a variety of ambiguous grammars you could have chosen from. A particularly short one is:

$$S \rightarrow S - S \mid a$$

Here are two parse trees for the string $a+a+a$



Notice that the question asked for parse trees, not derivations. They are closely related, but not the same thing.

One of the shortest solutions was a variant of this idea:

$$\begin{array}{l}
 S \rightarrow a \mid A \\
 A \rightarrow a
 \end{array}$$

Many people also got good mileage about variables that could optionally expand into the empty string.

6. (4 points) Define what it means for a function mapping strings to strings to be a homomorphism.

Solution: This turns out to have been a really hard question.

A function h on strings is a homomorphism if $h(x)h(y) = h(xy)$ for all strings x and y .

Or, equivalently, a function h is a homomorphism if its output on a string is the concatenation of its outputs on individual characters. That is, $h(c_1c_2 \dots c_n) = h(c_1)h(c_2) \dots h(c_n)$. for any string $c_1c_2 \dots c_n$.

Saying that a homomorphism maps each character to a string isn't a technically complete answer, but it captures much of the idea.

7. (4 points) We showed in class that the language $B = \{0^n1^n \mid n \geq 0\}$ is not regular. Let $L = \{2^i0^n1^n \mid n \geq 0, i \geq 0\}$. Prove that L is not regular using closure properties and the fact that B isn't regular.

Solution: Suppose that L were regular. Then $L \cap 0^*1^*$ would be regular, because the intersection of two regular languages is regular. But $L \cap 0^*1^*$ is just B , which we know not to be regular. This is a contradiction. So we must have been wrong in our assumption that L was regular.

Or, suppose that L were regular. Let h be a homomorphism which maps 2 to ϵ , and 0 and 1 to themselves. Then $h(L)$ must also be regular, because regular languages are closed under homomorphism. But $h(L)$ is B , which we know not to be regular. This is a contradiction. So we must have been wrong in our assumption that L was regular.

Many people wrote proofs which relied on the idea that applying some operation (e.g. intersection) to a regular language and a non-regular language yields a non-regular language. This isn't true. A correct answer to this kind of question normally requires a proof by contradiction.

Another common wrong approach involved claiming that $L \cap B$ must be regular because regular languages are closed under intersection. However, B isn't regular and the inputs to the closure properties must both be regular.

8. (4 points) Finish the following statement of the pumping lemma.

If A is a regular language, then there is a number p such that, if s is any string in A of length at least p , then ...

Solution: ... there are strings x , y and z such that

- $s = xyz$
- $|xy| \leq p$
- $|y| \geq 1$, and
- $xy^iz \in A$ for every $i \in \mathbb{N}$.

Each of the four conditions was worth 1/4 of the points.