CS173: Discrete Mathematical Structures, Spring 2008

Second Midterm Solutions

1. (10 points) Let $f(n) = 2n \log_2(2n)$ and let $g(n) = n \log_2 n^2$. Prove that $f(n) = \Theta(g(n))$.

   Solution: Using limits: $f(n) = 2n \log 2n$ can be written as $f(n) = 2n + 2n \log n$. Thus, $\lim_{n\to\infty} f(n)/g(n) = \lim_{n\to\infty}(2n + 2n\log n)/n\log n = \lim_{n\to\infty}(2/\log n) + (2) = 2$ and $0 < 2 < \infty$.

   Finding constants: Again any number of combinations of c, N will work. In particular for the big-O bound, let $c = 4$ and $N = 1$, giving $2n\log 2n = 2n + 2n\log n \leq 2nlogn + 2nlogn$ which is true for all $2n \leq 2n\log n$ or for $1 \leq \log n$. The big-omega lower bound is also simple: set $c = N = 1$, giving $2n + 2n\log n \geq n\log n$ or $2n + n\log n \geq 0$.

   Note: many different answers are possible. ■

2. (10 points) Find a recurrence for $f(n)$, the number of bitstrings of length n that do not have three consecutive ones. For example, $f(3) = 7$ because out of the 8 bitstrings of length 3, only one has three consecutive ones.

   $$\{000, 001, 010, 011, 100, 101, 110, \cancel{111}\}$$

   You do <u>not</u> have to solve the recurrence - just give the recurrence and explain your answer.

   Solution: Base cases: $f(0) = 1$, $f(1) = 2$, $f(2) = 4$

   Recurrence: $f(n) = f(n-1) + f(n-2) + f(n-3)$

   Explanation: Suppose $f(n) =$ the number of bitstrings with no three consecutive 1's. Consider the first bits of the bitstring. If we have a 0, there are $f(n-1)$ possible bitstrings with no three consecutive ones. If there is a leading 1, we either have a 0 or 1 next. There are $f(n-2)$ ways to begin with 10, since any bitstring with no three consecutive ones can come afterwards. If we have 11, then it must be followed by a 0, so there are $f(n-3)$ such bitstrings beginning with 110.

   Grader's note: Very few people got this question, despite the fact that it came directly from discussion section.

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3. (5 points per part, 10 points total) Give the general form for the following recurrences. Note that you do <u>not</u> have to solve for constants. For example, the general form for the recurrence $R(n) = 3R(n-1) + 5$ would be $R(n) = c_1 3^n + c_2$, where c_1 and c_2 are constants.

   (a) $A(n) = (-3)A(n-1) + 4 \cdot 5^n$

   Solution: The characteristic equation of the homogeneous part of the recurrence is $x + 3 = 0$, with a characteristic root of $x = -3$. This gives us one solution of $A(n) = c_1(-3)^n$. For the inhomogeneous part, we have $g(n) = 4 \cdot 5^n$, a polynomial of degree 0 multiplied by 5^n, where 5 is not a characteristic root. This gives us a solution of $A(n) = c_2 5^n$. Putting these together as a linear combination, we get:

   $$A(n) = c_1(-3)^n + c_2 5^n$$

   Rubric: 1 point for getting the right characteristic equation, 1 point for solving it correctly and writing $(-3)^n$, 1 point for giving a generic polynomial of degree 0, 1 point for writing 5^n, and 1 point for putting it all together as a linear combination. ■

(b) $B(n) = B(n-1) + 2B(n-2) + 2^n(n-2)$

Solution: The characteristic equation of the homogeneous part of the recurrence is $x^2 - x - 2 = 0$, with characteristic roots $x_1 = -1, x_2 = 2$. This gives us a solution of $B(n) = c_1(-1)^n + c_2 2^n$. For the inhomogeneous part, we have $g(n) = (n-2)2^n$, a polynomial of degree 1 multiplied by 2^n, where 2 is a characteristic root with multiplicity 1. This gives us a solution of $B(n) = n(c_3 + c_4 n)2^n$. Putting these together as a linear combination, we get:

$$B(n) = c_1(-1)^n + c_2 2^n + n(c_3 + c_4 n)2^n$$

Rubric: 1 point for getting the right characteristic equation, 1 point for solving it correctly and writing $(-1)^n$ and 2^n, 1 point for giving a generic polynomial of degree 1, 1 point for writing remembering to multiply by n to account for the root's multiplicity, and 1 point for putting it all together as a linear combination. ■

4. (10 points) Let R_1 and R_2 be relations on a set A. Prove or disprove: if R_1 and R_2 are transitive, then $R_1 \cap R_2$ is transitive.

Solution: Let $(a, b), (b, c) \in R_1 \cap R_2$. Then by the definition of intersection, $(a, b), (b, c) \in R_1$ and $(a, b), (b, c) \in R_2$. Thus since R_1 and R_2 are transitive, $(a, c) \in R_1$ and $(a, c) \in R_2$. Thus by the definition of intersection, $(a, c) \in R_1 \cap R_2$. Since, a, b, c were arbitrary, $R_1 \cap R_2$ is transitive, as desired.

Grader's note: Several students gave two transitive relations with empty intersection, or with intersection $\{(a, b)\}$. It is important to note that these relations *are* transitive, since there are no two pairs of the form $(a, b), (b, c)$ in them. Another common mistake was check that if $(a, b), (b, c) \in R_1, R_2$, then $(a, c) \in R_1 \cap R_2$. This is essentially correct, however, you need to start with arbitrary pairs in $R_1 \cap R_2$, and then argue that they must be in both R_1 and R_2, since you are proving that $R_1 \cap R_2$ is transitive. ■

5. Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be given by $f(m, n) = mn$.

(a) (5 points) Is f one-to-one? Prove your answer.

Solution: This function is not one-to-one. To see this, simply note that $f(0, 1) = 0 \cdot 1 = 0 = 0 \cdot 2 = f(0, 2)$, but $(0, 1) \neq (0, 2)$ ■

(b) (5 points) Is f onto? Prove your answer.

Solution: This function is onto. To see this, let $x \in \mathbb{Z}$ be arbitrary. Note that $(1, x) \in \mathbb{Z} \times \mathbb{Z}$, and that $f(1, x) = 1 \cdot x = x$. Since x was arbitrary, every element of the codomain $(\mathbb{Z})$ is an image of something in the domain, and so f is onto. ■

6. If S is a finite set of numbers, then $\max S$ denotes the largest element in S and $\min S$ denotes the smallest element in S. For each of the following relations on $A = \mathcal{P}(\{1, 2, \ldots, 10\})$, decide if the relation is reflexive, irreflexive, transitive, or antisymmetric. Each can satisfy more than one of these properties; circle all that apply. You do not need to prove your answers are correct.

(a) $R_1 = \{(S, T) \in A \times A : \max S \leq \max T\}$.

Solution: Reflexive and Transitive ■

(b) $R_3 = \{(S,T) \in A \times A : \max S \leq \min T\}$.

Solution: Transitive and Antisymmetric ■

(c) $R_4 = \{(S,T) \in A \times A : S \subseteq T \text{ or } T \subseteq S\}$.

Solution: Reflexive and Symmetric ■

Grader's Note: If you get nothing else from this problem, learn that it is possible for a relation to be neither symmetric nor antisymmetric; similarly, it is possible for a relation to be neither reflexive nor irreflexive (you should have seen this in discussion section).

7. (10 points) Let $T(n) = 6T(\frac{n}{6}) + n^2$, with $T(1) = 1$. Solve the recurrence $T(n)$ asymptotically. Show your work. You may use any theorems or methods presented in class. If you use the guess and check method, then you must prove that your answer is correct. If you use other methods from class, then you do not need to prove your answer is correct.

Solution: This could be done using either Master theorem or recursion trees.

Master theorem: Here, $a = 6$ and $b = 6$, so $af(n/b) = 6 \cdot (\frac{n}{6})^2 = \frac{n^2}{6}$. Since $f(n) = n^2$, this means $af(n/b) = \frac{1}{6}f(n)$, giving that $T(n) = \Theta(n^2)$.

Recursion tree: A node at level i of the recursion tree has $(\frac{n}{6^i})^2$, and there are 6^i nodes, so each level has total work $\frac{n^2}{6^i}$. There are $\log_6 n$ levels, giving $T(n) = \sum_{i=0}^{\log_6 n} \frac{n^2}{6^i}$. We then get $T(n) \leq n^2 \sum_{i=0}^{\infty} \frac{1}{6^i} = (6/5)n^2$, so $T(n) = O(n^2)$. Similarly, $T(n) \geq n^2 \sum_{i=0}^{0} \frac{1}{6^i} = n^2$, so $T(n) = \Omega(n^2)$. Since we have both upper and lower bounds, this gives $T(n) = \Theta(n^2)$.

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8. (15 points) Let $T(n) = \sqrt{n}T(\sqrt{n}) + n$, with $T(2) = 1$. Solve the recurrence $T(n)$ asymptotically. Show your work. You may use any theorems or methods presented in class. If you use the guess and check method, then you must prove that your answer is correct. If you use other methods from class, then you do not need to prove your answer is correct.

Solution: This problem is from lecture and from the recurrence handout offered on the website; please see those references for the solution.

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9. (10 points) Let A be a finite set, and $f : A \to A$ be a one-to-one function. Prove that f is also onto.

Solution: Use pigeonhole: let the "circles" be the elements in the codomain and the "dots" be the elements of the domain which are mapped to them. Let $|A| = n < \infty$, so we have n circles, and n dots. But we know that no two dots can be in the same circle, since the function is one-to-one. By the pigeonhole principle, we cannot have an empty circle, so every element in the codomain has an element mapping to it from the domain. In other words, the function is onto.

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