CS 173: Midterm Exam 1

Spring 2005

Name: SOLUTION

NetID: ______________________________

Lecture Section: ____________________

General Directions

1. Make sure your name is on every page.
2. There are 11 pages, including a sheet of scratch paper. Make sure that you answer all 14 questions.
3. Remember to write clearly and legibly. Unreadable answers will receive no credit.
4. This is a closed book exam. No notes of any kind are allowed.
5. Remember to time yourself.

Question	Points	Out of
1		5
2		5
3		5
4		5
5		5
6		5
7		5
8		5
9		10
10		10
11		10
12		10
13		10
14		10
Total		100

Multiple Choice

Problem 1 (5pts)

Which one of the following functions is a bijection from $\mathbf{R}^+$ to $\mathbf{R}^+$?

a) $f(x) = (x-1)^2$

b) $f(x) = e^x$

c) $f(x) = \log_2(x+1)$

d) $f(x) = \frac{1}{1+x^2}$

Solution

c

Problem 2 (5pts)

Which one of the following propositions is **NOT** a tautology?

a) $(p \wedge q \wedge \neg p) \rightarrow r$

b) $(p \vee q) \rightarrow (p \wedge \neg r)$

c) $(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$

d) $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

Solution

b

Problem 3 (5pts)

Let $A = \{\phi, \{\phi\}, \{\phi, \{\phi\}\}\}$.Which one of the following statements is true?

a) $\{\phi, \{\{\phi\}\}\}$ is in the power set of A.

b) $A \times \{\phi\} = \phi$.

c) Let $B = \{\phi, \{\phi\}\}$. Then $B \in A$ and $B \subseteq A$.

d) $A \cup \{\{\phi\}\} = \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$

Solution

c

Problem 4 (5pts)

A, B, and C are subsets of a set S. Which one of the following set identities is **NOT** true?

a) $A \cap (B \cup \overline{A}) = B \cap A$

b) $(A - B) - C = (A - C) - B$

c) $(A - B) - C = (A - C) - (B - C)$

d) $\overline{(\overline{A} \cup \overline{B}) \cap \overline{A}} = A \cap B$

Solution

d

Problem 5 (5pts)

Which one of the following expresses the negation of *"Everybody loves somebody sometime"*?

a) Everybody hates somebody sometime.

b) Somebody loves everybody all the time.

c) Somebody hates everybody all the time.

d) Everybody hates everybody all the time.

Solution

c

Problem 6 (5pts)

Which one of the following statements is **NOT** true?

a) If a function f is a *bijection* from set A to set B, there must be an inverse function f^{-1} such that $f^{-1}(b) = a$ when $f(a) = b, a \in A, b \in B$.

b) If a function is *onto*, then its codomain must be equal to its image.

c) A function must be either one-to-one, onto, or both.

d) Let $f : S \to T$ and $g : T \to U$. If $g \circ f$ is *one-to-one*, then f must be *one-to-one*.

Solution

c

Problem 7 (5pts)

Given the predicate symbols

$D(x)$ is "x is a dog."

$R(x)$ is "x is a rabbit."

$C(x, y)$ is "x chases y."

Which one of the following is an appropriate translation of the English statement *"Only dogs chase rabbits"*?

a) $(\forall y)(\exists x)((R(y) \wedge C(x, y)) \rightarrow D(x))$

b) $(\exists y)(\forall x)((R(y) \wedge C(x, y)) \rightarrow D(x))$

c) $(\forall y)(\forall x)((R(y) \wedge C(x, y)) \rightarrow D(x))$

d) $(\exists y)(\exists x)((R(y) \wedge C(x, y)) \rightarrow D(x))$

Solution

c

Problem 8 (5pts)

Which one of the following arguments is **NOT** valid?

a) If n is a real number with $n > 4$, then $n^2 > 16$. Suppose that $n^2 \leq 16$. Then $n \leq 4$.

b) If n is a real number with $n > 2$, then $n^2 > 4$. Suppose that $n \leq 2$. Then $n^2 \leq 4$.

c) All students in this class love CS173. Xavier is a student in this class. Therefore, Xavier loves CS173.

d) Every computer science major takes CS173. Natasha does not take CS173. Therefore, Natasha is not a computer science major.

Solution

b

Short Answer Problems

Problem 9 (10pts)

Prove the inference rule known as ”simplification” without using truth tables.

Solution

$p \wedge q \implies p \equiv \neg(p \wedge q) \vee p \equiv (\neg p \vee \neg q) \vee p \equiv (p \vee \neg p) \vee q \equiv T$

Comments

Each mistake drops two points.

Problem 10 (10pts)

For each of the following parts, describe a function mathematically. A picture is not sufficient.

a) Describe a function $f : \mathbf{Z} \to \mathbf{Z}$ which is one-to-one, but not onto.

b) Describe a function $f : \mathbf{Z} \to \mathbf{Z}$ which is onto, but not one-to-one.

c) Describe a function $f : \mathbf{Z} \to \mathbf{Z}$ which is bijective.

d) Describe a function $f : \mathbf{Z} \to \mathbf{Z}$ which is neither one-to-one, nor onto.

Solution

Some possible answers:

a) $f(x) = 2x$

b) $f(x) = \lfloor \frac{x}{2} \rfloor$

c) $f(x) = x$

d) $f(x) = 2$

Comments

Each mistake drops two points. The most common source of errors was not realizing what it means to have a function from $\mathbf{Z} \to \mathbf{Z}$. This includes using functions that are not defined everywhere for integers and assuming that a function which is onto for reals is also onto for integers.

Problem 11 (10pts)

Prove or disprove the following statements.

a) For any integers a and b, $a \mod 5 = b \mod 5 \rightarrow a^2 \mod 5 = b^2 \mod 5$.

b) For any integers a and b, $a^2 \mod 5 = b^2 \mod 5 \rightarrow a \mod 5 = b \mod 5$.

Solution

a) We first *prove* that $a \mod 5 = r = b \mod 5 \rightarrow a^2 \mod 5 = b^2 \mod 5$.

Let $a = 5k + r$, and $b = 5j + r$, with $r \in \{0, 1, ..., 4\}$. (Any integer can be written in this form.)
Then $a^2 = 25k^2 + 10rk + r^2$, and $b^2 = 25j^2 + 10rj + r^2$.
Thus, $a^2 = 5(5k^2 + 2rk) + r^2 = r^2 \mod 5$. Similarly, $b^2 = r^2 \mod 5$.

b) We *disprove* that $a^2 \mod 5 = b^2 \mod 5 \rightarrow a \mod 5 = b \mod 5$.

All that is needed is a counterexample. Let $a = 3, b = 2$. Then, $a^2 \mod 5 = b^2 \mod 5 = 4$. However, $a \neq b$.

Comments

a) 5 points.
1 point for $a = 5k + r$.
1 point for $b = 5j + r$.
1 point for squaring.
1 point for factoring.
1 point for the argument.

b) 5 points.
1 point for recognizing that the claim is false.
4 points for the counterexample.

Problem 12 (10pts)

Prove that if $S \subseteq T$, then $S \cup T = T$.

Solution

The proof can be done directly or indirectly, but similar intricacies exist in each.
Assume $S \subseteq T$.
First, show $S \cup T \subseteq T$.
Choose $x \in S \cup T$. Then, $x \in S$ or $x \in T$ by definition of union.
Case 1, $x \in S$. Then $x \in T$ since $S \subseteq T$. Case 2, $x \in T$.

Second, show $T \subseteq S \cup T$. Choose $x \in T$. Then $x \in S \cup T$ by addition.

Comments

10 points.
2 points for a reasonable attempt.
1 point each for writing down correct definitions of $\subseteq$, and $\cup$.
1 point for assuming $S \subseteq T$.
2 points each for $S \cup T \subseteq T$, and vice versa.
1 point for cases.

Problem 13 (10pts)

100 students are sick. 1000 students are tired. 10000 students are antsy for spring break. Determine the number of students who are sick or tired or antsy for spring break under each of the following conditions:

a) Every sick student is tired, and every tired student is antsy for spring break.

b) The sets of students are pairwise disjoint.

c) There are two students who are both sick and tired, two who are both tired and antsy for spring break, two who are both sick and antsy for spring break, and one who is sick and tired and antsy for spring break.

Solution

a) 10000

b) 11100

c)

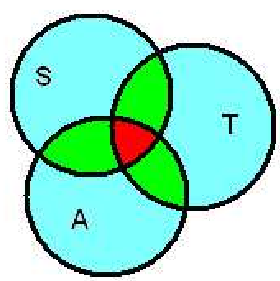
$$\begin{aligned} & |S \cup T \cup A| \\ = & \ |S| + |T| + |A| - |S \cup T| - |T \cup A| - |S \cup A| + |S \cap T \cap A| \\ = & \ 100 + 1000 + 10000 - 2 - 2 - 2 + 1 \\ = & \ 11095 \end{aligned}$$

Comments

a) 3 points
This one is saying that set S (students that are sick) is a subset of T and T is a subset of A. So $S \cup T \cup A$ is just A. Therefore, $|S \cup T \cup A| = |A|$.

b) 3 points
Pairwise disjoint means the every pair of sets does not have overlapping part. Therefore, $|S \cup T \cup A| = |S| + |T| + |A|$.

c) 4 points
This is a standard problem of set operations.
First, we add up $|S|$, $|T|$, and $|A|$. Since the green parts (i.e. $|S \cap T|$, $|T \cap A|$, and $|S \cap A|$) are added twice



and the red part ($|S \cap T \cap A|$) is added three times, we next subtract duplicate ones. After subtracting $|S \cap T|$, $|T \cap A|$, and $|S \cap A|$ from the sum, we find that $|S \cap T \cap A|$ is actually subtracted three times. So we finally add $|S \cap T \cap A|$ once. Therefore, the overall process looks like:

$$|S \cup T \cup A| = |S| + |T| + |A| - |S \cup T| - |T \cup A| - |S \cup A| + |S \cap T \cap A|$$

Problem 14 (10pts)

Let $P(x)$, $Q(x)$, and $R(x)$ be the statements "x is a duck," "x is annoying," and "x is a dancer," respectively. Express each of these statements using quantifiers, logical connectives, and $P(x)$, $Q(x)$, and $R(x)$.

a) All ducks are annoying.

b) Some dancers are not annoying.

c) Some dancers are not ducks.

d) Does c) follow from a) and b)? Prove your response.

Solution

a) $\forall x(P(x) \to Q(x))$

b) $\exists x(R(x) \land \neg Q(x))$

c) $\exists x(R(x) \land \neg P(x))$

d) (c) follows from (a) and (b).

(1)	$P(c) \to Q(c)$	*universal instantiation from* (a)
(2)	$R(c) \land \neg Q(c)$	*existential instantiation from* (b)
(3)	$\neg Q(c)$	*simplification from* (2)
(4)	$\neg P(c)$	*Modus Pollens from* (1) *and* (3)
(5)	$R(c)$	*simplification from* (2)
(6)	$R(c) \land \neg P(c)$	*conjunction from* (4)*and* (5)
(7)	$\exists x(R(x) \land \neg P(x))$	*existential generalization form* (6)

Comments

a) 2 points.
Note that $\forall x(P(x) \land Q(x))$ is not correct. It means "everything" is a duck and it is annoying.

b) 2 points.
Note that $\exists x(R(x) \to \neg Q(x))$ is not correct. It is true even when $R(x)$ is false, which is obviously not what we want.

c) 2 points.
Note that $\exists x(R(x) \to \neg P(x))$ is not correct. The reason is the same as above.

d) 4 points. Techniques other than rules of inference are also acceptable. But you need to **clearly** state your reasoning.