CS 173: Midterm Exam II Solutions

Spring 2004

Name: ______________________________

NetID: ______________________________

Lecture Section: ____________________

General Directions

1. Make sure your name is on every page.
2. There are 7 pages. Make sure that you answer all 13 questions.
3. Remember to write clearly and legibly. Unreadable answers will receive no credit.
4. This is a closed book exam. No notes of any kind are allowed.
5. Remember to time yourself.

Question	Points	Out of
1		6
2		6
3		6
4		6
5		6
6		6
7		6
8		6
9		10
10		10
11		10
12		10
13		12
Total		100

Multiple Choice

Problem 1 (6pts)

When sorted in increasing order of growth rate, which one of the following functions would be second?

a) $n^3 + n - 8$

b) $\log n + n$

c) $n \log n$

d) $\frac{n!}{7}$

Solution

(c) is second. The growth rate in these problems can be determined by two basic rules:

1. Theorem: Suppose that $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$. Then $(f_1+f_2)(x)$ is $O(\max(|g_1(x)|, |g_2(x)|))$.
2. Theorem: Suppose that $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$. Then $(f_1 f_2)(x)$ is $O(g_1(x)g_2(x))$.

Applying our rules, we can work the problem as follows:

a) $n^3 + n - 8 = O(\max(n^3, n, 1)) = O(n^3)$

b) $\log n + n = O(\max(\log n, n)) = O(n)$

c) $n \log n = O((n)(\log n)) = O(n \log n)$

d) $\frac{n!}{7} = O((n!)(1)) = O(n!)$

$O(n) < O(n \log n) < O(n^3) < O(n!)$

Problem 2 (6pts)

Which of the following equalities describe(s) the relation between functions $f(x) = (n + \frac{2}{n})(\log n + n^3)$ and $g(x) = n^3 \log n$? (Choose all that apply)

a) $f(x) = \Omega(g(x))$

b) $g(x) = \Omega(f(x))$

c) $f(x) = O(g(x))$

d) $g(x) = O(f(x))$

Solution

(a) and **(d)**.

$$f(x) = (n + \frac{2}{n})(\log n + n^3) = O((n)(n^3)) = O(n^4)$$

n^4 always grows faster than $n^3 \log n$, so we can find constants to satisfy **(d)** $g(x) = O(f(x))$. Then, if we consider the definition of big-O and big-Ω, we realize that if $g(x) = O(f(x))$, then **(a)** $f(x) = \Omega(g(x))$. This is because $g(x) = O(f(x))$ means there are constants c, k such that $|g(x)| \leq c|f(x)|$ for $x > k$, which we can manipulate to get $|f(x)| \geq \frac{1}{c}|g(x)|$ for $x > k$, which fits the definition of $f(x) = \Omega(g(x))$.

Problem 3 (6pts)

f is a mapping from the set of positive integers to real numbers. $f(x) = x^2 + \sqrt{x} - 3$. Which of the following is a/are correct observation(s)?

a) f is one-to-one.

b) f is onto.

c) f is invertible.

d) None of the above can be concluded.

Solution

(a) is the only correct observation.

It is important to consider the following definitions:

1. $f(x)$ is one-to-one if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f.
2. $f(x)$ is onto if and only if for every element b in the codomain, there is an element a in the domain such that $f(a) = b$.
3. $f(x)$ is invertible if it is bot one-to-one and onto.

a) f is one-to-one. This is true because f is a mapping from the set of positive integers to real numbers. Therefore, the overlapping influence of the x^2 term is avoided, and each input produces a unique output.

b) f is onto. This is false because in order to do so every single one of the reals would have to have some positive integer map to it.

c) f is invertible. In order for this to be true, both (a) and (b) would have to be true. Since (b) is false, f is not invertible.

d) None of the above can be concluded. We concluded (a), so (d) does not apply.

Problem 4 (6pts)

Given $f_1 = 1$, $f_n = \frac{f_{n-1}}{n+1}$, f_5 is:

a) $\frac{3}{8}$

b) $\frac{1}{12}$

c) $\frac{5}{12}$

d) $\frac{1}{360}$

Solution

(d) is correct.

$f_1 = 1$

$f_2 = \frac{f_1}{2+1} = \frac{1}{3}$

$f_3 = \frac{f_2}{3+1} = \frac{\frac{1}{3}}{4} = \frac{1}{12}$

$f_4 = \frac{f_3}{4+1} = \frac{\frac{1}{12}}{5} = \frac{1}{60}$

$f_5 = \frac{f_4}{5+1} = \frac{\frac{1}{60}}{6} = \frac{1}{360}$

Problem 5 (6pts)

The set S is defined by $4 \in S$ and $s \cdot t \in S$, whenever $s \in S$ and $t \in S$. Which of the following elements does NOT belong to S? (Choose all that apply)

a) 16

b) 32

c) 64

d) 256

Solution

(b) does not belong to S.

All the elements of S can constructed using 4, which happens to be 2^2. Thus, all elements of S must have the form 2^{2n}, where n is a positive integer.

a) $16 = 2^4$. Alternatively, we can construct it using $4 \cdot 4$.

b) $32 = 2^5$. The exponent is odd, so it cannot belong to S.

c) $64 = 2^6$. Using our result from (a), we can construct it using $4 \cdot 16$.

d) $256 = 2^8$. Using our result from (a), we can construct it using $16 \cdot 16$.

Problem 6 (6pts)

The function $f(x) = x \log(x^2)$ is big-O of which of the following functions? (Choose all that apply)

a) $g(x) = x$

b) $g(x) = x^2$

c) $g(x) = x^2 \log x$

d) $g(x) = x \log x$

Solution

(b), (c), (d) are all correct.

$f(x) = x \log(x^2) = x(2 \log x) = 2x \log x = O(x \log x)$. Furthermore, if $f(x) = O(x \log x)$, then it is also O of any function that grows faster than $x \log x$. This means that f is also $O(x^2)$ and $O(x^2 \log x)$.

Problem 7 (6pts)

Which of the following sets is/are countable?

a) The set of integers.

b) The set of real numbers.

c) The set of rationals.

d) The set of natural numbers.

Solution

(a), (c), (d) are all countable.

a) The set of integers. We can count these by mapping the non-negative integers to the even natural numbers, and the negative integers to the odd numbers.

b) The set of real numbers. The set of reals is uncountable by Cantor's Diagonalization method.

c) The set of rationals. The set of rationals is countable by the chart constructed on page 235, shown below without guiding arrows.

$$\begin{array}{cccccc} \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} & \cdots \\ \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \cdots \\ \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \cdots \\ \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4} & \cdots \\ \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \frac{5}{5} & \cdots \end{array}$$

d) The set of natural numbers. We determine if a set S is infinitely countable by constructing a bijection from the set S to the natural numbers. Of course, we're able to contruct a bijection from $\mathbf{N}$ to $\mathbf{N}$.

Problem 8 (6pts)

Determine which of the following functions is a bijection from $\mathbf{R}$ to $\mathbf{R}$. (Choose all that apply)

a) $f(x) = 2x + 1$

b) $f(x) = x^2 + 1$

c) $f(x) = \frac{\log x}{x+1}$

d) $f(x) = x^3$

Solution

(a) and **(d)** are both bijections from $\mathbf{R}$ to $\mathbf{R}$.

a) $f(x) = 2x + 1$. $f(x)$ is one-to-one, meaning that $f(a) = f(b)$ if and only if $a = b$. We can show this by

$$\begin{aligned} 2a + 1 &= 2b + 1 \\ 2a &= 2b \\ a &= b \end{aligned}$$

Likewise, $f(x)$ is onto, if we choose $a = \frac{b-1}{2}$.

$$\begin{aligned} f(a) &= f(\tfrac{b-1}{2}) \\ &= 2(\tfrac{b-1}{2}) + 1 \\ &= (b-1) + 1 \\ &= b \end{aligned}$$

Thus $f(x)$ is a bijection.

b) $f(x) = x^2 + 1$. $f(x)$ is not one-to-one, because both x and $-x$ map to the same value.

c) $f(x) = \frac{\log x}{x+1}$. $f(x)$ is not a valid function, since negative inputs have undefined outputs.

d) $f(x) = x^3$. $f(x)$ is one-to-one and onto, so it is a bijection.

Short Answer Problems

Problem 9 (10pts)

Use the pigeonhole principle to argue that any set of 10 nonempty strings over $\{a, b, c\}$ have two different strings whose starting letters agree and ending letters agree.

Solution

Nonempty strings over $\{a, b, c\}$ can be any size greater than 0, can use any combination of letters, and can have any order. Fortunately, we are only concerned with the starting and ending letters of the strings. There are 9 possible combinations of starting and ending letters for strings. We use * to denote the interior of the string. We also note that if a string is of length 1, then it's single letter is both the starting and ending letter.

1. a*a
2. a*b
3. a*c
4. b*a
5. b*b
6. b*c
7. c*a
8. c*b
9. c*c

Since there are only 9 possible combinations, and 10 strings, by the pigeonhole principle, at least two of the strings must have the same starting and ending letter.

Problem 10 (10pts)

Let $A_1, A_2, \ldots$ be a sequence of countable sets. Show that $\bigcup_{i=1}^{\infty} A_i$ is countable.

Solution

We can show that $\bigcup_{i=1}^{\infty} A_i$ is countable by constructing a bijection from the set to the natural numbers. We can do this by enumerating the members of the sets as follows:

$A_1 = \{a_{11}, a_{12}, a_{13}, a_{14}, ...\}$ $A_2 = \{a_{21}, a_{22}, a_{23}, a_{24}, ...\}$ ...

Then we can construct a bijection like the one used to count the rationals on page 235 of the textbook:

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{14} & \ldots \\ a_{21} & a_{22} & a_{23} & a_{24} & \ldots \\ a_{31} & a_{32} & a_{33} & a_{34} & \ldots \\ a_{41} & a_{42} & a_{43} & a_{44} & \ldots \\ \ldots & & & & \end{array}$$

Problem 11 (10pts)

Show that if f and g are real-valued functions such that $f(x) = O(g(x))$, then $f^k(x) = O(g^k(x))$. (Note: $f^k(x)$ is defined as the kth power of $f(x)$).

Solution

We are given that $f(x) = O(g(x))$, meaning there are constants c_1 and c_2 such that $|f(x)| \leq c_1|g(x)|$ for all $x > c_2$. Considering this equation, we can raise both sides to the kth power to get $|f^k(x)| \leq |c_1^k g^k(x)|$ for all $x > c_2$. Since c_1^k and c_2 are constants, this means that $f^k(x) = O(g^k(x))$.

Problem 12 (10pts)

Let X be the set of strings over $\{a, b\}$ of length 4 and let Y be the set of strings over $\{a, b\}$ of length 3. Define the functions $f : X \to Y$ and $g : Y \to Y$ by the following rules:

$$f(x) = \text{the string consisting of the first 3 characters of } x$$
$$g(y) = y^R \text{ (the reversal of } y\text{)}$$

Answer the following questions and justify your responses:

a) is f one-to-one?

b) is f onto?

c) is g one-to-one?

d) is g onto?

e) is $g(f(x))$ one-to-one?

f) is $g(f(x))$ onto?

Solution

a) **No**. As a counterexample, f will return aba for both $abaa$ and $abab$.

b) **Yes**. Every string in Y comprises the first 3 letters of at least one string in X, so it is onto. Suppose $S \in Y$, then $Sa \in X$ and $Sb \in X$ will both map to it.

c) **Yes**. If $g(a) = g(b)$, then

$$\begin{aligned} a^R &= b^R \\ (a^R)^R &= (b^R)^R \\ a &= b \end{aligned}$$

d) **Yes**. Suppose we have an arbitrary element $a \in Y$. Then $g(a) = a^R$ is also an element of Y. But then we have $g(a^R) = (a^R)^R = a$, so every element in Y has an element that maps to it.

e) **No**. As a counterexample, $abba$ and $abbb$ will both map to bba.

f) **Yes**. f is onto, and so is g. This means that f will map to all elements of Y, and g will map to all elements of Y. Suppose we have an element $c \in Y$. Then there is an element $d \in Y$ that will map to it, since g is onto. Furthermore, there is an element $e \in X$ that will map to d under f, since f is onto. Therefore $g(f(x))$ is onto.

Long Problem

Problem 13 (12pts)

We will use mathematical induction to show that a $2n \times 3n$ board can be completely covered using L-shaped tiles.

a) Show by induction that the proposition holds for $2 \times 3n$ and $3 \times 2n$ boards. (Hint: Both of these proofs should be very short.)

b) Using part(a) and induction, show that the original proposition is correct.

Solution

a) Basis step: Combine two tiles to get a 2×3 and 3×2 board.

Inductive Step: Assume the inductive hypothesis, that $2 \times 3n$ and $2n \times 3$ boards can be tiled. We wish to prove then that $2 \times 3(n+1)$ and $3 \times 2(n+1)$ boards can be tiled. We can do this by realizing that a $2 \times 3(n+1)$ board is really a $2 \times 3n$ space with an extra 2×3 space on the end. We can tile the $2 \times 3n$ space by the inductive hypothesis, and we can tile the remaining 2×3 space with two tiles. Therefore a $2 \times 3(n+1)$ board is tileable. Likewise, a $3 \times 2(n+1)$ board is a $3 \times 2n$ space with a 3×2 space at the end. Once again, the $3 \times 2n$ space is tileable by the inductive hypothesis, and we can tile the remaining 3×2 space with two tiles. Therefore a $3 \times 2(n+1)$ board is tileable.

b) Base Case: From part (a), we know that both $2 \times 3n$ and $3 \times 2n$ boards are tileable, and a 2×3 board is tileable.

Inductive Step: Assume the inductive hypothesis, that a $2n \times 3n$ boards can be tiled. We then wish to prove that $2(n+1) \times 3(n+1)$ boards are tileable. To show this, we note that a $2(n+1) \times 3(n+1)$ board is a $2n \times 3n$ space, a $2n \times 3$ space, a $3n \times 2$ space, and a 2×3 space. The $2n \times 3n$ space is tileable using the inductive hypothesis. Furthermore, the $2n \times 3$ space and the $3n \times 2$ space are tileable as base cases. Finally, the 2×3 space is tileable using two tiles. Therefore a $2n \times 3n$ board is tileable.