University of Illinois at Urbana-Champaign
Department of Computer Science

Final Exam

CS 173 Discrete Mathematics
Spring, 2004

Friday, May 7, 2004

Print your name neatly in the space provided below, and sign in the area provided. Do **not** place your social security number anywhere on this exam.

Name:

Netid:

Lecture Section:

Signature:

This is a **closed book** and **closed notes** exam.
Try to do ALL problems in this booklet. Read each question very carefully. If you think that there is an ambiguous statement, make an explicit assumption, and keep on solving using your assumption. If you run out of time, at least try to explain what you are trying to do.

You should have 14 pages. Page numbered 14 is draft paper. You may remove it but you have to turn in the draft paper at the end. You can also use the back sides of the pages as draft area of area for the solution, if needed.

Question	Points	Score	Grader
1	6		
2	6		
3	6		
4	6		
5	6		
6	6		
7	6		
8	6		
9	6		
10	6		
11	10		
12	10		
13	10		
14	10		
15	10		
16	10		
17	15		
18	15		
Total	150		

You must return all pages of this exam.

Multiple Choice (choose all that apply)

Problem 1

Which of the following quantifications is/are true?

a) $\forall x \in \mathbb{R}\ (x^2 \neq -1)$

b) $\exists x \in \mathbb{Z}\ (x^2 = 2)$

c) $\forall x \in \mathbb{Z}\ (x^2 > 2)$

d) $\exists x \in \mathbb{R}\ (x^2 = x)$

Problem 2

Which of the following statements can be concluded about the sets A and B if $A \oplus B = A$?

a) $A = \emptyset$

b) $B = \emptyset$

c) $A = B$

d) $A = \bar{B}$

Problem 3

Let the proposition $p \downarrow q$ be true when both p and q are false, and false otherwise. Which one of the following propositions is equivalent to $p \to q$?

a) $(p \downarrow p) \downarrow (q \downarrow q)$

b) $(p \downarrow q) \downarrow (p \downarrow q)$

c) $((p \downarrow p) \downarrow q) \downarrow ((p \downarrow p) \downarrow q)$

d) $((p \downarrow q) \downarrow p) \downarrow ((q \downarrow p) \downarrow q)$

Problem 4

Which of the following functions has the slowest order of growth?

a) $\log n!$

b) $n^{\log n}$

c) $\log \log n$

d) $n(\log n)^2$

Problem 5

What is the running time of the following algorithm? (Choose all that apply.)

```
for i = 1 to n
    j := n;
    while j >= 1
       j := j / 2;
```

a) $O(\log n)$

b) $O(n)$

c) $O(n \log n)$

d) $O(n^2)$

Problem 6

Given that $f_4 = 4$, $f_5 = 7$, $f_6 = 13$, which of the following recursive definitions could have generated these terms? Choose all that apply.

a) $f_1 = 1, f_2 = 2$, and $f_n = f_{n-1} + 2f_{n-2}$ when $n > 2$.

b) $f_1 = 1, f_2 = 3$, and $f_n = f_{n-1} + f_{n-2}$ when $n > 2$.

c) $f_1 = 1, f_2 = 1, f_3 = 1$, and $f_n = f_{n-1} + 2f_{n-2} + f_{n-3}$ when $n > 3$.

d) $f_1 = 1, f_2 = 1, f_3 = 1$, and $f_n = f_{n-1} + f_{n-2} + 2f_{n-3}$ when $n > 3$.

Problem 7

Let R be a relation on the set $A = \{a, b, c, d\}$. If $R = \{(a, a), (b, b), (c, c), (d, d)\}$, then R has which of the following properties? (Choose the *best* answer. Be *very* careful!)

a) reflexive

b) reflexive and symmetric

c) reflexive, symmetric and transitive

d) reflexive, symmetric, antisymmetric and transitive

Problem 8

The symmetric closure of a relation drawn from a set with cardinality n will require (Choose all that apply. Think of the matrix representation of the the relation.)

a) at least n elements to be added to the original relation

b) at most $\frac{n^2-n}{2}$ elements to be added to the original relation

c) as few as zero elements to be added to the original relation

d) at most $\frac{n}{2}$ elements to be added to the original relation

Problem 9

$P = (\{1, 2, 4, 6, 12\}, |)$ is a poset where '|' means 'divides'. Which of the following statements are true regarding P? (Choose all that apply.)

a) 4 and 6 are incomparable

b) P has a least element

c) 6 and 12 are the maximal elements of P

d) P has a greatest element

Problem 10

Let G be a directed graph with 25 edges and 100 vertices. The sum of the in-degrees and out-degrees of the vertices of G is:

a) 0

b) 25

c) 12

d) 50

Short Answer Problems

Problem 11 (10pts)

Express the negations of each of these statements so that all negation symbols immediately precede predicates.

a) $\exists z \forall y \forall x \; T(x, y, z)$

b) $\exists x \exists y \; P(x, y) \wedge \forall x \forall y Q(x, y)$

c) $\exists x \exists y \; (Q(x, y) \rightarrow Q(y, x))$

Score:

Problem 12 (10pts)

The *fibonacci* sequence is defined as:

$$f_1 = 1$$
$$f_2 = 2$$
$$f_n = f_{n-1} + f_{n-2}, n \geq 3.$$

Use mathematical induction to prove that for $n \geq 5$,

$$f_n > \left(\frac{3}{2}\right)^n$$

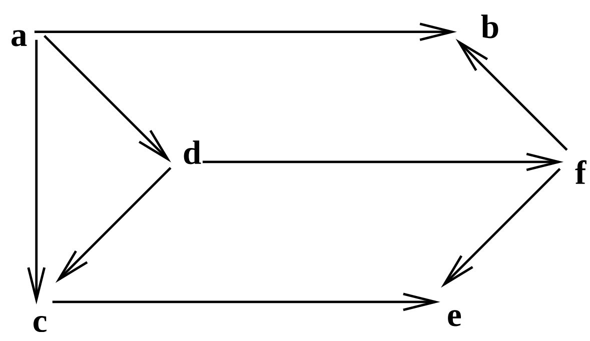
Problem 13 (10pts)

Let R be a relation such that $R^2 \subseteq R$. Use the definitions of transitivity and R^2 to show that R is transitive.

Problem 14 (10pts)

Let X be a non-empty set. Define a relation on $P(X)$, the power set of X, as $(A, B) \in R$ if and only if $A \subseteq B$. Is this relation reflexive, symmetric, antisymmetric, transitive and/or a partial order? Justify your answer.

Score:



Problem 15 (10pts)

The above directed graph represents a relation R.

a) List the edges that need to be added to the above graph to form the directed graph representing the transitive closure of R.

b) Let R' be the transitive closure of the reflexive closure of R. Notice R' is a poset. Draw its Hasse diagram.

c) What is/are the minimal element(s) of R'?

d) What is/are the maximal element(s)?

e) What is the least element (if any)?

f) What is the greatest element (if any)?

g) What are the upper bounds of the set $\{b, e\}$?

h) What are the lower bounds of the set $\{b, c\}$?

Problem 16 (10pts)

Consider a polynomial of degree n with integer coefficients: $a_n x^n + a_{n-1} x^{n-1} + ... + a_0$, where $a_0, a_1, ..., a_n \in \mathbf{Z}$ and $n \in \mathbf{Z}^+$.

a) Show that the set of all polynomials of degree 1 (n=1) are countable.

b) Using part (a) as the base case, prove by induction that the set of all polynomials of a fixed degree n are countable.

Score:

Long Problems

Problem 17 (15pts)

The *ternary* search algorithm searches for an element x in a sorted list as follows. At each step it first compares the element at position $n/3$ in the list to x and then possibly compares the element at position $2n/3$ in the list to x. In every step, it either discovers x and terminates or reduces the size of the list to one-third its current size.

a) Given a single element x and a list of n elements that contains x, where n is a power of 3, devise a recursive algorithm for this search strategy to find the position of x in the list.

b) Analyze the time complexity of your algorithm.

Problem 18 (15pts)

Definition: If f is a function with domain A, then the relation $\sim$ defined by

$$x \sim y \text{ iff } f(x) = f(y)$$

is an equivalence relation on A, and it is called the *kernel relation* of f.

An interesting consequence of equivalence relations and partitions is that any function f can be factored into a composition of two functions, one an injection, and one a surjection. For a function $f : A \to B$, let P be a partition of A by the kernel relation of f. Then define the function $s : A \to P$ by $s(a) = [a]$ and define $i : P \to B$ by $i([a]) = f(a)$.

a) Prove that s is a surjection.

b) Prove that i is an injection.

c) Prove that $f = i \circ s$.

Draft Paper