Name: ____________________ Net ID: ____________________ Univ. ID # ____________________

# Midterm Exam 1, CS 173, Spring 2003
## Tuesday, February 25, 2003
## SOLUTIONS

**Read the following instructions before you start to work.**

- **This is a closed-book, closed-notes exam!** No calculators, paper, notebooks, or anything else. If you brought anything with you other than pencils, erasers, or pens, please leave them at the front of the room with the proctor.
- Print your name, Net-ID, and University ID in the slots above, and print your name at the top of every page.
- Write your answers in the spaces provided. You can use the backs of sheets for scratch. See the proctor if you need more scratch paper.
- Read each question carefully and make sure you understand exactly what it is asking. If you give a beautiful answer to the wrong question, you'll get no credit. Simplify your answers as much as possible. Correct but unnecessarily complicated answers might not receive full credit.
- For reasons of consistency, the proctors will only answer syntactic questions about the exam (e.g. is that an "a" or an "e"). If you think the question is ambiguous, make and state your reasonable assumptions and answer the question.
- Don't spend too much time on any single problem. You have 105 minutes, and there are 110 points on the exam, so if a problem looks like it will take you much longer in minutes than its point value, you might leave it for later. Remember, *this exam is curved.*
- Ambiguously marked answers will receive no credit.
- Points for each problem are indicated.
- For True/False questions, your score is the number correctly marked minus the number incorrectly marked. For all other problem types there is no penalty for incorrect answers.
- Make sure you have all 8 pages.

1. [4 points] Write each of the following statements in the form "If ... Then ...".

(a) $x$ is even only if $y$ is odd.
**Solution:** If $x$ is even then $y$ is odd.

(b) It is hot whenever it is sunny.
**Solution:** If it is sunny then it is hot.

(c) Studying is sufficient for passing.
**Solution:** If you study then you pass

(d) Studying is necessary for passing.
**Solution:** If you pass then you studied

2. [5 points] Write each of the following compound propositions using logical connectives and symbols, where $c$, $d$, $r$, and $w$, mean "it is cold", "it is dry", "it is rainy", and "it is windy", respectively.

(a) It is neither cold nor dry.
**Solution:** $\neg c \wedge \neg d$. (Or: $\neg(c \vee d)$.)

(b) it is rainy if it is not dry.
**Solution:** $\neg d \rightarrow r$.

(c) To be cold, it must be windy or rainy.
**Solution:** $c \rightarrow (w \vee r)$

(d) It is cold or dry, but not both.
**Solution:** $c \oplus d$ (Or $(c \vee d) \wedge (\neg c \vee \neg d)$)

(e) Unless it is windy and dry, it is raining.
**Solution:** There is some disagreement about what "A unless B" means. One interpretation is that if B holds, A does not. The other interpretation is the same, but adds also that if B doesn't hold, then A does. So, "unless $B$, $A$" can be written either as $\neg B \rightarrow A$, or (second interpretation) $\neg B \leftrightarrow A$ (which is the same as $A \oplus B$).
Acceptable answers:
$\neg(w \wedge d) \rightarrow r$
$\neg(w \wedge d) \leftrightarrow r$
$(w \wedge d) \oplus r$.

3. [5 points] Write the truth table for $\neg(r \to \neg q) \vee (p \wedge \neg r)$. Be careful, there is no partial credit.

| $p$ | $q$ | $r$ | $\neg(r \to \neg q) \vee (p \wedge \neg r)$ |
|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $T$ |
| $T$ | $F$ | $T$ | $F$ |
| $T$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $F$ |
| $F$ | $F$ | $F$ | $F$ |

4. [4 points] Find a compound proposition that has the following truth table.

| $p$ | $q$ | ? |
|---|---|---|
| $T$ | $T$ | $F$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $F$ |

**Solution:** $\neg p \wedge q$, or $\neg(q \to p)$, or other simple equivalent expressions.

5. [4 points] Find a compound proposition with the above truth table, but that uses only the operators $\vee$ and $\neg$.
**Solution:** $\neg(p \vee \neg q)$

6. [6 points] For each of the following, either explain why it is a tautology or give a counterexample showing that it is not.

(a) $((p \to q) \wedge \neg p) \to \neg q$
**Solution:** Not a tautology: is false when $p$ false and $q$ true.

(b) $((p \to \neg q) \wedge q) \to \neg p$
**Solution:** Tautology; many explanations possible. Example: left side says that $\neg p \vee \neg q$, but also $q$. Thus $\neg p$.

7. [4 points] For each of the following, if the pair of expressions are logically equivalent, circle "T", otherwise circle "F".

| | | | |
|---|---|---|---|
| (a) **Solution: True** | $(p \oplus q)$ | –and– | $\neg(p \leftrightarrow q)$ |
| (b) **Solution: False** | $\neg(p \leftrightarrow q)$ | –and– | $\neg(p \rightarrow q)$ |
| (c) **Solution: True** | $p \rightarrow q$ | –and– | $\neg q \rightarrow \neg p$ |
| (d) **Solution: True** | $\neg p \leftrightarrow q$ | –and– | $p \leftrightarrow \neg q$ |
| (e) **Solution: False** | $p$ | -and- | $q$ |
| (f) **Solution: False** | $p \rightarrow (q \rightarrow r)$ | –and– | $p \rightarrow (q \wedge r)$ |
| (g) **Solution: False** | $p \vee (q \wedge r)$ | –and– | $(p \wedge q) \vee (p \wedge r)$ |
| (h) **Solution: True** | $p \rightarrow (\neg q \wedge r)$ | –and– | $\neg p \vee \neg(r \rightarrow q)$ |

8. [6 points] Match each of the following with the correct English statement by writing the letter next to the number. (DO NOT draw "matching" lines.)

| | | |
|---|---|---|
| **(b)** ______ | (1) $\exists x \forall y T(x,y)$. | (a) Every course has at least one student. |
| **(d)** ______ | (2) $\exists y \forall x T(x,y)$. | (b) Some student is enrolled in every course. |
| **(e)** ______ | (3) $\forall x \exists y T(x,y)$. | (c) No student is taking every course. |
| **(i)** ______ | (4) $\neg \exists x \exists y T(x,y)$. | (d) Some course is being taken by all students. |
| **(g)** ______ | (5) $\exists x \forall y \neg T(x,y)$. | (e) Every student is taking at least one course. |
| **(a)** ______ | (6) $\forall y \exists x T(x,y)$. | (f) Some course has an enrollment of 0. |
| **(f)** ______ | (7) $\exists y \forall x \neg T(x,y)$. | (g) Some students are not taking any courses. |
| **(g)** ______ | (8) $\neg \forall x \exists y T(x,y)$. | (h) No course is being taken by all students. |
| **(h)** ______ | (9) $\neg \exists y \forall x T(x,y)$. | (i) No student is taking any course. |
| **(b)** ______ | (10) $\neg \forall x \exists y \neg T(x,y)$. | (j) All students are having a party. |
| **(g)** ______ | (11) $\neg \forall x \neg \forall y \neg T(x,y)$. | (k) The rain in Spain falls mainly on the plain. |
| **(c)** ______ | (12) $\forall x \exists y \neg T(x,y)$. | (l) No students choose this answer. |

9. [10 points] Prove that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are equivalent using logical equivalences.

**Solution:** $\neg(p \vee (\neg p \wedge q))$
$= \neg p \wedge \neg(\neg p \wedge q))$ by DeMorgan
$= \neg p \wedge (p \vee \neg q))$ by DeMorgan again
$= \neg p \wedge p) \vee (\neg p \wedge \neg q)$ by distributive law
$= F \vee (\neg p \wedge \neg q)$ since $p \wedge \neg p = F$.
$= \neg p \wedge \neg q$ since $F \vee x = x$.

10. [10 points] Suppose the variable $x$ has domain of discourse the set of all students, and $y$ the set of all courses, and we define the following predicates:
$U(y)$ : $y$ is an upper-level course.
$C(y)$ : $y$ is a CS course.
$F(x)$ : $x$ is a freshman.
$A(x)$ : $x$ is a part-time student.
$B(x)$ : $x$ is a full-time student.
$T(x, y)$ : student $x$ is taking course $y$.

Write each of the following statements using the above predicates and any needed quantifiers:

(a) Harvey is taking CS 173.
**Solution:** $T(\text{Harvey, CS173})$

(b) Not all freshman are full-time students.
**Solution:** $\exists x F(x) \wedge \neg B(x)$, OR $\neg(\forall x(F(x) \rightarrow B(x)))$. OR $\exists x \neg(F(x) \rightarrow B(x))$.
Note however that $\forall x \neg(F(x) \rightarrow B(x))$ is incorrect.

(c) No CS course is upper-level.
**Solution:** $\neg\exists y(C(y) \wedge U(y))$ OR $\forall y(C(y) \rightarrow \neg U(y))$.

(d) There is a part-time student who is not taking any CS course.
**Solution:** $\exists x \forall y A(x) \wedge (C(y) \rightarrow \neg T(x, y))$. OR $\exists x(A(x) \wedge \forall y(C(y) \rightarrow \neg T(x, y)))$

(e) Every part-time freshman is taking some upper-level CS course.
**Solution:** $\forall x(A(x) \wedge F(x)) \rightarrow \exists y(U(y) \wedge C(y) \wedge T(x, y))$.
OR $\forall x \exists y A(x) \wedge F(x) \rightarrow U(y) \wedge C(y) \wedge T(x, y)$.

11. [1 point] Find the sets $A$ and $B$ if $B - A = \{2, 10\}$, $A - B = \{1, 5, 7, 8\}$, and $A \cap B = \{3, 6, 9\}$.
**Solution:** $A = \{1, 3, 5, 6, 7, 8, 9\}$, $B = \{2, 3, 6, 9, 10\}$

12. [4 points] Denote the empty set by $\emptyset$. What is the Power-set of $\{\emptyset, 1, \{1\}\}$?

**Solution:** $\{\emptyset, \{\emptyset\}, \{1\}, \{\{1\}\}, \{\emptyset, 1\}, \{\emptyset, \{1\}\}, \{1, \{1\}\}, \{\emptyset, 1, \{1\}\}\}$

13. [5 points] Let $A = \{a, b, c\}$. Mark each of the following True (T) or False (F).

(a) **True** $\{b, c\} \in P(A)$.

(b) **True** $\{\{a\}\} \subseteq P(A)$.

(c) **True** $\{\emptyset\} \subseteq P(A)$.

(d) **True** $\emptyset \subseteq A \times A$.

(e) **False** $\{a, c\} \in A \times A$.

14. [10 points] Prove or disprove: for all sets $A, B, C$, $\quad A - (B \cap C) = (A - B) \cup (A - C)$.
Do not use Venn Diagrams in your proof.
**Solution:** It is true. We can prove it using logical equivalences, or English argument.

**Proof 1:** We'll show that the left side is a subset of the right, and vice versa. To show the left a subset of the right, consider an arbitrary element $x \in A - (B \cap C)$. Thus $x$ is in $A$, but $x$ is not in the intersection of $B$ and $C$. If $x$ isn't in the intersection of $B$ and $C$, then either $x$ is not in $B$, or $x$ is not in $C$ (or both). So either $x$ is in $A$ but not $B$, or $x$ is in $A$ but not $C$ (or both). In other words, $x \in (A - B) \cup (A - C)$.

Now to see that the right is a subset of the left, consider any $x \in (A - B) \cup (A - C)$. Thus either $x \in A - B$ or $x \in A - C$ (or both). So, either $x$ is not in $B$ or $x$ is not in $C$ (or both), and in either case, $x$ is not in the intersection of $B$ and $C$. Hence, $x \in A$, but not $B \cap C$, so $x \in A - (B \cap C)$

**Proof 2:** Use logical equivalences:
$A - (B \cap C) = A \cap \neg(B \cap C)$ by definition of set difference
$= A \cap (\neg B \cup \neg C)$ by DeMorgan's law
$= (A \cap \neg B) \cup (A \cap \neg C)$ by the distributive law
$= (A - B) \cup (A - C)$ by the definition of set difference.

15. [6 points] For each argument, say whether or not it is valid.

**(a) Given:**
$p \rightarrow r$
$q \rightarrow r$
$\neg(p \vee q)$
**Therefore:** $\neg r$
**Solution:** Not valid. $r$ could be true even if both $p$ and $q$ are false, without violating any of the statements.

**(b) Given:**
She is a Math Major or a Computer Science Major.
If she does not know discrete math, she is not a Math Major.
If she knows discrete math, she is smart.
She is not a Computer Science Major.
**Therefore:** she is smart.
**Solution:** Valid. The fourth and first lines allow us to deduce that she is a math major. The contrapositive of the second statement lets us then conclude that she knows discrete math, and together with the third statement (and modus ponens) she must be smart.

**(c) Given:**
It is not rainy and it is not cold.
If it is not July, then it is rainy or cold.
If it is July or August, then it is rainy.
**Therefore:** It is rainy.
**Solution:** Valid! The first statement, together with the contrapositive of the second, implies that it is not July. By the third statement, it is rainy. (Of course, this contradicts the first statement. This shows that the three premises taken together are logically inconsistent. In fact, from such a collection, we can deduce anything (how?).

16. [8 points] A large sum of money has been stolen from a bank. The criminals were seen driving away from the scene. Afterwards, Artie, Bernie, Charlie, and Dufus were questioned. Suppose the following facts have been established:

   (a) Artie is afraid of spiders.

   (b) Bernie does not know how to drive.

   (c) Charlie never pulls a job without using both Artie and Dufus.

   (d) Dufus never works with less than two other people.

   (e) No one other than Artie, Bernie, Charlie, and Dufus was involved in the robbery.

   Prove by contradiction that Artie must be guilty. Your proof should be a sequence of statements, and for each statement you make you should explain exactly why it follows from previous statements and the facts above, which you should refer to by letter.

   **Solution:** Assume that Artie is not guilty.
   By (c), Charlie cannot be guilty, since he must work with Artie if he is guilty.
   By (e), the only remaining possibilities are Bernie and Dufus.
   By (d), Dufus is not guilty, since he needs two accomplices, and only one (Bernie) remains.
   This leaves only Bernie, who by (b) could not have done it alone.
   So, if Artie is not guilty, then nobody is, but this contradicts the fact that a crime was committed and (e).

   Other arguments are possible, but a bit more verbose. If you didn't do a proof by contradiction, then you had a harder time solving the problem, and you didn't get full credit (max 7, if everything else was correct.)

   We can also do this symbolically, though that is not what was asked for. Moreover, it is not easy to do that way completely. (You'd have to introduce predicates $D(x)$ for "$x$ drives", and $G(x)$ for "$x$ is guilty". The last statement would also give you trouble.)

17. [20 points]

A diamond cutter cuts $n$-carat diamonds into $n$ single 1-carat diamonds. Whenever he makes a cut, the diamond is split into two smaller diamonds with a whole number of carats (no fractions). He is paid for each cut as follows: When a diamond is cut into two pieces, he receives a number of dollars equal to the product of the size in carats of the resulting two pieces. For example, if he cuts a 9-carat diamond into a 4 and a 5-carat diamond, he'd make 4 times 5, or 20 dollars for that single cut. If he then cuts the 5-carat diamond into a 2-carat and 3-carat diamond, he'd make 6 dollars more. He continues to split the pieces (currently 4-carats, 3-carats, and 2-carats) in any manner, collecting money for each cut, until there are 9 1-carat diamonds left.

A rather counterintuitive fact: no matter how he does this, the total amount of money that he makes for splitting up an $n$-carat diamond into $n$ 1-carat diamonds is always the same: $n(n-1)/2$ dollars. For example, regardless of how he cuts up a 9-carat diamond, he will make 36 dollars.

Prove by induction that this is indeed true: No matter what sequence of cuts is used to cut an $n$-carat diamond into $n$ 1-carat diamonds, the total amount of money made will always be $n(n-1)/2$ dollars, assuming as above that the amount paid for a single cut is the product of the resulting sizes.

**Solution:**

**Base Case:** A 1-carat diamond earns 0 dollars, and $0 = 1(1-1)/2$.

**Inductive Step:** Assume inductively that for any $k$-carat diamond with $k < n$, the amount of money made regardless of how it is cut is exactly $k(k-1)/2$. Now consider an $n$-carat diamond. The first cut must divide the diamond into two pieces, one of size $k$, the other of size $n-k$, for some $k$ such that $1 < k < n$ and $1 < n-k < n$. The amount of money made by the first cut is thus $k(n-k)$. By the inductive hypothesis, regardless of how the two pieces are cut, the total they yield for the work are $k(k-1)/2$ and $(n-k)(n-k-1)/2$, respectively. Thus, the total amount of money made is: $k(n-k) + k(k-1)/2 + (n-k)(n-k-1)/2$. This simplifies to $n(n-1)/2$ (which we'd expect you to show).