CS 173: Midterm Exam 1

Fall 2005

Name: ______________________________

NetID: ______________________________

Lecture Section: ____________________

Section Leader: ____________________

General Directions

1. Make sure your name is on every page.
2. Remember to write clearly and legibly. Unreadable answers will receive no credit.
3. This is a closed book exam. No notes of any kind are allowed. No calculators.
4. Remember to time yourself.

Question	Points	Out of
1		5
2		5
3		5
4		5
5		5
6		5
7		5
8		5
9		10
10		10
11		10
12		10
13		20
Total		100

Multiple Choice

Problem 1 (5pts)

Which of the following is logically equivalent to $p \to q$?

a) $q \vee \neg p$

b) the contrapositive of $p \to q$

c) the inverse of the converse of $p \to q$

d) all of the above

Answer: (d).

Problem 2 (5pts)

Which of the following is a negation of $\forall x \forall y[((x > 0) \wedge (y > 0)) \to (x + y > 0)]$?

a) $\exists x \exists y[(x > 0) \wedge (y > 0) \wedge (x + y \leq 0)]$

b) $\exists x \exists y[((x \leq 0) \vee (y \leq 0)) \wedge (x + y > 0)]$

c) $\forall x \forall y \neg[((x > 0) \wedge (y > 0)) \to (x + y > 0)]$

d) $\exists x \exists y[(x \leq 0) \vee (y \leq 0) \vee (x + y > 0)]$

Answer: (a).

Problem 3 (5pts)

Which of the following must by true?

a) $(X \cap Y) \cup (Y - X) = X$

b) $X \times (Y - Z) = (X \times Y) - (X \times Z)$

c) $X - (Y \cup Z) = (X - Y) \cup Z$

d) If $A - C = B - C$ then $A = B$.

Answer: (b).

Problem 4 (5pts)

Which of the following is not a tautology? Hint: You can answer this fastest if you think about inference rules.

a) $[\neg q \wedge (p \rightarrow q)] \rightarrow \neg p$

b) $(p \wedge q) \rightarrow p$

c) $[(p \vee q) \wedge \neg p] \rightarrow q$

d) $[(p \vee q) \wedge (\neg p \vee r)] \rightarrow q \wedge r$

Answer: (d).

Problem 5 (5pts)

If all sets are finite, which of the following must be true?

a) If a function is bijective, its domain and co-domain have the same cardinality.

b) If a function is one-to-one, its domain and co-domain have the same cardinality.

c) If a function is onto, its domain and co-domain have the same cardinality.

d) If a function is neither one-to-one nor onto, its domain and co-domain do not have the same cardinality.

Answer: (a).

Problem 6 (5pts)

Which of the following arguments is valid?

a) $\forall x(S(x) \rightarrow L(x)), S(a) \therefore L(a)$

b) $\forall x(S(x) \rightarrow L(x)), L(a) \therefore S(a)$

c) $\forall x(S(x) \rightarrow L(x)), \neg S(a) \therefore \neg L(a)$

d) $\exists x(S(x) \wedge L(x)), S(a) \therefore L(a)$

Answer: (a).

Problem 7 (5pts)

Let $f(x) = 3x + 2$ and $g(x) = x^2$ be functions defined on the integers ($f : Z \rightarrow Z$, $g : Z \rightarrow Z$). Which of the following is true?

a) $g \circ f = O(x^2)$

b) $g \circ f = O(x^3)$, and $g \circ f$ is not $O(x^2)$

c) $g \circ f(x) = f \circ g(x)$

d) $g \circ f$ has an inverse function.

Answer: (a).

Problem 8 (5pts)

Which of the following is false?

a) $\{x\} \subseteq \{x\}$

b) $\{x\} \in \{x, \{x\}\}$

c) $\{x\} \subseteq \mathcal{P}(\{x\})$, where $\mathcal{P}(\{x\})$ is the power set of $\{x\}$

d) $\{x\} \subseteq \{x, \{x\}\}$

Answer: (c).

Short Answer Problems

Problem 9 (10pts)

Use an indirect proof for the following:

Given: $p \to (m \to w)$

$w \to d$

m

$\neg d$

Prove: $\neg p$

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1.  Assume p           (since we must have an indirect proof)
2.  p->(m->w)          Given
3.  m->w               modus ponens (1,2)
4.  w->d               Given
5.  m->d               Hypothetical Syllogism (3,4)
6.  m                  Given
7.  d                  modus ponens (5,6)

(7) is a contradiction of the given hypothesis (~d).  Since
the negation of the conclusion led to a contradiction, the
original conclusion (~p) is proven.

Grading rubric:

2 pts. for assuming the opposite (p)
1 pt.  for using "Given" statements
2,2 pts. for each modus ponens step (should have been used twice)
2 pts. for a correct conclusion (arriving at a contradiction)
1 pt. for labelling each step.
Total:  10 pts.

If a student used a direct proof, we took off 3 pts.
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Problem 10 (10pts)

Prove that $B - A \subseteq \overline{A - B}$, for any sets A and B.

Solution: First observe that

$$B - A = B \cap \overline{A} \tag{1}$$

$$\overline{A - B} = \overline{A \cap \overline{B}} = \overline{A} \cup B = B \cup \overline{A}. \tag{2}$$

We have to show that $\forall x \colon x \in B \cap \overline{A} \rightarrow x \in B \cup \overline{A}$. Let a be an arbitrary element in $B \cap \overline{A}$. By definition, a is in B and it is in $\overline{A}$. Hence, a is in B. Since $B \subseteq B \cup \overline{A}$, the element a must be in $B \cup \overline{A}$. The element a was arbitrary; thus, the claimed set inclusion holds.

Problem 11 (10pts)

Tell whether each of the following is True or False. The universe is all integers.

a) $\forall z \forall y \exists x (x - y = z)$

Solution

True

b) $\forall y \exists x \exists z (x - y = z)$

Solution

True

c) $\forall x \forall y \forall z (x - y = z)$

Solution

False. This is saying for all pairs of (x, y, z), it holds that $x - y = z$. Obviously, it is false.

d) $\forall x \forall y \exists z (x - y = z)$

Solution

True

e) $\forall x \exists y \exists z (x - y = z)$

Solution

True

f) $\exists x \exists y \forall z (x - y = z)$

Solution

False. This is saying, there exists a pair (x, y) such that $x - y = z$ holds for any z. We know that for any fixed pair of (x, y), their difference is also fixed. Therefore, it is false.

g) $\exists x \exists y \exists z (x - y = z)$

Solution

True

h) $\exists x \forall y \forall z (x - y = z)$

Solution

False. It is saying there exists a particular x such that $x - y = z$ holds for any pair of (y, z).

Criteria

-1 point for each incorrect answer.

Problem 12 (10pts)

Prove that $8n^2 + n$ is $O(\frac{n^2}{2} - 5)$.

Solution

We want to prove that there exist k,c such that, for any $n \geq k$, $8n^2 + n < c(n^2/2 - 5)$.
Set $c = 18$. We get $8n^2 + n < 18(n^2/2 - 5)$. Thus
$8n^2 + n < 9n^2 - 90$.
$n^2 - n - 90 > 0$
$(n - 10)(n + 9) > 0$.
The roots are $-9, 10$. If we select $n > 10$, we see that the quadratic inequality holds (it's a good idea to show the parabolla graph as in solving quadratic inequality).

So $c = 18, k = 11$. End of proof.

Criteria

8/10-9/10: for minor errors.
5/10-7/10: for not solving the quadratic inequality (depends on how well argued the case was).
2/10-5/10: for trying to use limits, growth rates, etc (depends whether you wrote the definition of big-O or not).
2/10: if all you remembered was the definition of Big-O, but didn't apply it on the problem.
1/10: for trying to do anything (except just copying the problem formulation) and for the sense of humor.

Problem 13 (20pts)

a) Let j and k be integers, with j even and k odd. Prove that the product of j and k is even.

Solution

$j = 2n$, and $k = 2m + 1$ for some integers n, m. Then $jk = 2n(2m + 1) = 2z$ for some integer z. Thus, the product jk is even.

Criteria

6 points for perfect answer. -2 points for not distinguishing m and n. -1 not concluding argument with the definition of an even number. -4 points for an answer that only specifies j and k, with no further argument. -5 points for an effort in the wrong direction.

b) Prove that the product of consecutive integers is even. Hint: you can use part a) in your solution.

Solution

Every pair of consecutive integers has one even and one odd. By part a) the product of such a pair is even.

Criteria

7 points for perfect answer. -3 for restricting the least value to be even or odd. -1 point for other minor errors.

c) Prove that the square of an odd integer equals $8k + 1$ for some integer k. Hint: you can use part b) in your solution.

Solution

We wish to consider the square of an odd number: $(2n + 1)^2$. $(2n + 1)^2 = 4n^2 + 4n + 1 = 4n(n + 1) + 1$. Notice that from part b), $n(n + 1)$ is even, so we have $(2n + 1)^2 = 4(2k) + 1$ for some integer k.

Criteria

7 points for perfect answer. 2 points awarded for setting up problem and multiplying out the square without further argument. 4 points awarded for correct setup, multiplication, and factoring.

SCRATCH PAPER