CS 173: Final Exam

Fall 2005

Name: _____

NetID: _____

Section Leader: _____

General Directions

1. Make sure your name is on every page.
2. There are 12 pages, including a sheet of scratch paper. Make sure that you answer all 17 questions.
3. Remember to write clearly and legibly. Unreadable answers will receive no credit.
4. This is a closed book exam. No notes of any kind are allowed.
5. Remember to time yourself.

Question	Points	Out of
1		5
2		5
3		5
4		5
5		5
6		5
7		5
8		5
9		5
10		5
11		5
12		5
13		20
14		20
15		20
16		15
17		15
Total		150

Multiple Choice

Problem 1 (5pts)

One of these things is not like the others. Choose the response whose truth value is different than the other three.

a) $\forall x \exists y (x + y = x)$

b) $\forall x \exists y (x + y = 0)$

c) $\exists y \forall x (x + y = x)$

d) $\exists y \forall x (x + y = 0)$

Problem 2 (5pts)

Define the set A to be $A = \{x, \{x\}, \{\{x\}\}, \{\{\{x\}\}\} \ldots\}$, and denote the power set of A by $P(A)$. Which of the following is an appropriate choice to fill in the blank:

$$A - \{x\} \_\_\_\_ P(A)$$

a) $=$

b) $\in$ (and not $\subseteq$)

c) $\subseteq$ (and not $\in$)

d) $\in$ or $\subseteq$

Problem 3 (5pts)

Suppose sets A and B are countably infinite. Which of the following is not necessarily true?

a) $A \cap B$ is countably infinite

b) $A \cup B$ is countably infinite

c) $|A| = |Z|$ where Z is the set of all integers

d) $|A| = |B|$

Problem 4 (5pts)

Let A be an $n \times n$ matrix, and suppose $A[i, j]$ contains the value in row i column j of the matrix. Assume that the value of $A[i, j]$ can be accessed in constant time.

Which of the following is not an upper bound for the running time of the following algorithm?

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Input: matrix A.

sum = 0
for k = 1 to n
    for j = k to n
            sum = sum + A[j, k]
    end for
end for return sum
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a) $O(n^2)$

b) $O(n^2 \log n)$

c) $O(n(n+1)/2)$

d) $O(n \log n)$

Problem 5 (5pts)

A shipment from Amazon.com contains 6 items, where each item is either a CD, DVD, or book. How many different types of shipments are possible?

a) 3^6

b) $\binom{8}{2}$

c) 6^3

d) 120

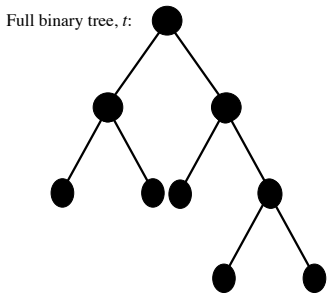
Problem 6 (5pts)

Let T be the set of full binary trees, and Z be the set of all integers. Define a recursive function $m : T \to Z$ to be:

$$m(t) = 0 \text{ if } t \text{ is a single vertex, and}$$

$$m(t) = 1 + \max\{m(t_1), m(t_2)\},$$

where t_1 and t_2 are the left and right subtrees of t, respectively.



Which of the following is $m(t)$ for tree t?

a) 3

b) 5

c) 6

d) 9

Problem 7 (5pts)

Suppose you know you will get exactly 60 or 70 or 80 points on an exam, and you know you are twice as likely to receive an 80 as a 70, and three times as likely to receive a 70 as a 60. What is your expected score on the exam?

a) 65

b) 70

c) 75

d) None of the above.

Problem 8 (5pts)

Suppose the running time of algorithm A is given by $T_n = 2T_{\frac{n}{2}} + O(n), T(1) = O(1)$. Which of the following gives the tightest upper bound on the running time of A? (Try to do this by reasoning, rather than by solving the recurrence exactly.)

a) $O(\log n)$

b) $O(n)$

c) $O(n \log n)$

d) $O(n^2)$

Problem 9 (5pts)

A group of students plans to order pizza. If 13 will eat sausage, 10 will eat pepperoni, 12 will eat olives, 4 will eat both sausage and pepperoni, 5 will eat pepperoni and olives, 7 will eat sausage and olives, and 3 will eat all three toppings, how many students are in the group?

a) 12

b) 22

c) 35

d) None of the above.

Problem 10 (5pts)

Consider the equivalence relation V on integers given by xVy if and only if $x + y$ is even. Which of the following best describes the set of equivalence classes of V?

a) $[1], [2]$

b) $[1], [2], [3], [4], \ldots$

c) $[2], [4], [6], [8], \ldots$

d) $[y]$, where y is any integer.

Problem 11 (5pts)

Let S be an n element set. How many different binary relations can be defined on S?

a) n^2

b) 2^n

c) $\binom{2^n}{2}$

d) 2^{n^2}

Problem 12 (5pts)

One of these things is not like the others. Determine which of the following graphs actually exist, and choose as your answer the one for which your response is different from the other three.

a) A graph of four vertices with degrees 1, 2, 3, and 4

b) A simple graph of four vertices with degrees 1, 2, 3, and 4

c) A connected graph with degrees 2, 3, 3, 4

d) A simple, connected, planar, graph with degrees 3, 3, 3, 3

Long Answer Problems

Problem 13 (20pts)

Suppose a set A has $n \geq 2$ elements and a set B has 2 elements. In this problem, we will count the number of functions $f : A \to B$ that are surjective (onto), and the number of functions $g : B \to A$ that are injective (one-to-one).

a) How many different functions f are there from A to B?

b) Describe the functions that are not onto.

c) How man functions are there of the type you described in part b?

d) How many onto functions, $f : A \to B$, are there?

e) How many different functions g are there from B to A?

f) How many one-to-one functions, $g : B \to A$, are there?

Problem 14 (20pts)

We define a **run** to be a maximal sequence of successes in a sequence of Bernoulli trials. For example, in the sequence $S, S, S, F, S, S, F, F, S$ (where S represents success and F represents failure) there are three runs consisting of three successes, two successes, and one success, respectively. Let R denote the random variable on the set of sequences of n independent Bernoulli trials that counts the **number of runs** in the sequence. Find $E[R]$, the expected number of runs in such a sequence if the probability of success on each Bernoulli trial is p, by answering the following questions.

To begin, define random variables I_j, $j = 1, \ldots, n$ so that $I_j = 1$ if a run **begins** at the jth Bernoulli trial, and $I_j = 0$, otherwise.

a) What is the probability that $I_1 = 1$?

b) If $I_j = 1$, what do you know about the $j - 1$st and jth Bernoulli trials?

c) What is the probability that $I_j = 1$ for $j = 2, \ldots, n$?

d) Write an expression for R, the number of runs, in terms of the I_j.

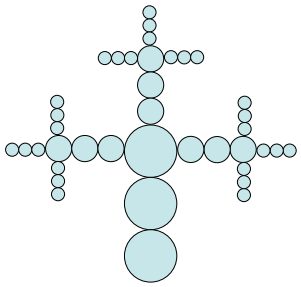
e) Find $E[I_1]$ using the definition of expectation of a random variable, and part a.

f) Find $E[I_j]$, where $j = 2, \ldots, n$, using the definition of expectation, and part c.

g) Use linearity of expectation, together with parts d, e, and f, to find $E[R]$.

Problem 15 (20pts)

Suppose a Flaky Snow Person of size k, $FSP(k)$, is a stack of 3 snow balls with 3 hairs on its head, each of which is a Flaky Snow Person of size $k-1$, $FSP(k-1)$. For example, here is a picture of $FSP(3)$:



a) Add 81 very small snowballs to the picture above to draw $FSP(4)$.

b) Write a recurrence for f_n, the number of snow balls in $FSP(n)$.

c) Compute f_2.

d) Solve the recurrence you wrote in part b.

e) Prove your recurrence is correct by induction.

Problem 16 (15pts)

a) Draw the Hasse diagram for the partial ordering "x divides y" on the set $\{3, 6, 9, 18, 54, 72, 108, 162\}$.

b) Name the minimum element, if it exists.

c) Name the minimal elements, if any exist.

d) Name the maximum element, if it exists.

e) Name the maximal elements, if any exist.

f) Give all lower bounds for the set $\{72, 108\}$, if any exist.

g) Give the greatest lower bound for the set $\{72, 108\}$, if it exists.

Problem 17 (15pts)

Solve the following recurrence:

$$a_n = 5a_{\frac{n}{2}} - 6a_{\frac{n}{4}} + \frac{n}{4}$$
$$a_1 = 3, a_2 = 9$$

SCRATCH PAPER