CS 173: Midterm Exam 2

Fall 2004

Name: ______________________________

NetID: ______________________________

Lecture Section: ____________________

General Directions

1. Make sure your name is on every page.
2. There are 10 pages, including a sheet of scratch paper. Make sure that you answer all 13 questions.
3. Remember to write clearly and legibly. Unreadable answers will receive no credit.
4. This is a closed book exam. No notes of any kind are allowed.
5. Remember to time yourself.

Question	Points	Out of
1		5
2		5
3		5
4		5
5		5
6		5
7		5
8		5
9		10
10		10
11		10
12		10
13		20
Total		100

Multiple Choice

Problem 1 (5pts)

Given the inductive definition: $f(1) = 2$, $f(2) = 2$, and $f(n) = 2f(n-1) + f(n-2)$ for $n > 2$, $f(5)$ is:

a) 8

b) 14

c) 34

d) 36

Problem 2 (5pts)

S is a collection of strings of symbols. It is recursively defined by

1. a and b belong to S.
2. if string X belongs to S, so does Xb.

Which of the following does **NOT** belong to S?

a) $abbb$

b) bbb

c) ba

d) a

Problem 3 (5pts)

Which of the following set is **uncountable**?

a) The set of real numbers between 172 and 173.

b) The set of integers

c) The set of integers not divisible by 3.

d) The union of two countable sets.

Problem 4 (5pts)

The function $f(x) = x^2 \log(x^2 + 100)$ is **big-**O of which of the following functions?

a) x^2

b) $x^2 \log x$

c) $x(\log x)^2$

d) $x \log x$

Problem 5 (5pts)

Let $f(x)$, $g(x)$, and $h(x)$ be functions, which of the following statement is **WRONG**?

a) If $f(x)$ is $\Theta(g(x))$ and $g(x)$ is $\Theta(h(x))$, then $f(x)$ is $\Theta(h(x))$.

b) If $f(x)$ is $O(g(x))$ and $g(x)$ is $O(h(x))$, then $f(x)$ is $O(h(x))$.

c) If $f(x)$ is $\Theta(g(x))$, then $g(x)$ is $\Theta(f(x))$.

d) If $f(x)$ is $O(g(x))$, then $g(x)$ is $O(f(x))$.

Problem 6 (5pts)

The **harmonic numbers** H_j, $j = 1, 2, 3, ...$, are defined by

$$H_j = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{j}.$$

Let n and m be two natural numbers, which of the following is equal to the expression:

$$2(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}) + \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{n+m}?$$

a) $H_{n+m} + H_m$

b) $H_{n+m} + H_n$

c) $H_{n+m} - H_m$

d) $H_{n+m} - H_n$

Problem 7 (5pts)

When sorted in **decreasing** order of growth rate, which one of the following functions would be **first**?

a) $(2n+1)(1+\log n^3)$

b) $4n+6n^2\log n$

c) $\frac{(n^2+1)(n^2+3)(n^2+5)}{n(n^2+2)(n^2+4)}$

d) $3n^2+2n\log(n^2+1)$

Problem 8 (5pts)

How many numbers must be selected from the set $\{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$ in order to guarantee that **at least** one pair adds up to 22?

a) 5

b) 6

c) 7

d) 8

Short Answer Problems

Problem 9 (10pts)

We say that a circle is a *positive integer circle* if it is centered at $(0, n)$ and has radius r, where n and r are positive integers. Show that the set of *positive integer circles* is countably infinite.

a) Using an indirect proof, show that the set of *positive integer circles* is infinite.

b) Prove that the set of *positive integer circles* is countable by defining a 1 to 1 function from the set of circles to the set of natural numbers.

Problem 10 (10pts)

Use the definition of big-O to show that for any positive integer constants a and b, $(n+a)^b = \Theta(n^b)$.

You may use the binomial theorem:
$(x+y)^n = \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} x^i y^{n-i}$

Problem 11 (10pts)

Prove using induction that every convex polygon with $n \geq 3$ vertices has exactly $\frac{n(n-3)}{2}$ diagonals.
(A convex polygon is a polygon with the property that every line segment drawn between any two points inside the polygon lies entirely inside the polygon.)

Problem 12 (10pts)

Prove that in any set of 11 integers, there are two whose difference is divisible by 10.

Long Problem

Problem 13 (20pts)

A certain set S of integers is defined recursively by the following rules:
1. 0 is in S
2. If i is in S, then $i + 5$ and $i + 7$ are in S.

a) What is the largest integer *not* in S? (8 points)

b) Let j be your answer to part a). Prove that all integers $j + 1$ and greater are in S. (12 points) (*HINT:* Use induction with a basis consisting of the integers $j + 1$ through $j + 5$.)

SCRATCH PAPER