Complexity Test 1 (makeup)

March 5, 2007

This is a closed-book, closed-notes, individual-effort test. Each problem is worth the same amount, but not all are of equal difficulty. Please write your name on each sheet of your solutions.

Problem 1:

Recall that $\mathbf{\Delta}^p_{k+1} = \mathbf{P}^{\mathbf{\Sigma}^p_k}$. Show that $\mathbf{\Sigma}^p_k \cup \mathbf{\Pi}^p_k \subseteq \mathbf{\Delta}^p_{k+1} \subseteq \mathbf{\Sigma}^p_{k+1} \cap \mathbf{\Pi}^p_{k+1}$.

Problem 2:

Define $\mathbf{DP}_k$ as follows:

$$\mathbf{DP}_0 = \mathbf{P}$$

$$\mathbf{DP}_{k+1} = \begin{cases} \{L \cup L' \mid L \in \mathbf{DP}_k, L' \in \mathbf{NP}\} & \text{if } k \text{ is even} \\ \{L \cap L' \mid L \in \mathbf{DP}_k, L' \in \textbf{co-NP}\} & \text{if } k \text{ is odd} \end{cases}$$

(a) Describe $\mathbf{DP}_1$ and $\mathbf{DP}_2$ explicitly in terms of $\mathbf{P}$, $\mathbf{NP}$ and $\textbf{co-NP}$. (Be sure to give the simplest expressions you can find.)

(b) Show that $\mathbf{DP}_k \subseteq \mathbf{DP}_{k+1}$ for all k.

(c) Show that $\mathbf{DP}_k \subseteq \mathbf{\Delta}^p_2$ for all k.

Problem 3:

Recall $\mathsf{PATH} = \{(G, s, t) \mid G$ is a directed graph with a path from s to $t\}$. Define a related language $\mathsf{ALLPATHS} = \{G \mid G$ is a directed graph with a path from every vertex to every other vertex $\}$.

(a) Show that $\mathsf{ALLPATHS}$ is in $\mathbf{NL}$.

(b) Show that in fact $\mathsf{ALLPATHS}$ is $\mathbf{NL}$-complete (w.r.t logspace reductions).

Problem 4:

Define computation of a (possibly non-boolean) function by a non-deterministic TM as follows: an NTM M is said to compute $f(x)$ if on input x, all threads of execution which do not output `abort` output $f(x)$, and at least one thread does not output `abort`.

Consider a relaxation of polynomial time reduction, called *non-deterministic* polynomial time reduction. It is a many-one reduction, in which the reduction is computed (as defined above) by a polynomial time non-deterministic TM.

Show that if a language in $\mathbf{P}$ is $\mathbf{NP}$-complete with respect to non-deterministic polynomial time reductions, then $\mathbf{NP} = \textbf{co-NP}$.

Problem 5:

Show that the following problem is in $\mathbf{\Sigma}^p_2$:

$$\mathsf{MAXSAT} = \left\{ (\phi, k) \;\middle|\; \begin{array}{c} \phi \text{ is a 3SAT formula and } k \text{ is (exactly) the greatest} \\ \text{number of clauses simultaneously satisfiable in } \phi \end{array} \right\}$$