Lecture 1

Projection Methods

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Projectors

- A projector maps to itself:
$$P^2 = P$$
- Then $(I-P)^2 = I - 2P + P^2 = I - P$
- Given $x \in \mathcal{N}(P)$, $(I-P)x = x$, so $x \in \mathcal{R}(I-P)$. Conversely, given $y \in \mathcal{R}(I-P)$, then $y = (I-P)y$ which mean $y \in \mathcal{N}(P)$.
$$\mathcal{N}(P) = \mathcal{R}(I-P)$$
- Any x can be *decomposed* into
$$x = Px + (I-P)x$$
- So
$$\mathbb{C}^n = \mathcal{N}(P) \oplus \mathcal{R}(P)$$

Projection Template

```
while  ||r|| > tol
   Generate  V = [v_1 ... v_m]  of  𝒦
   Generate  W = [v_1 ... v_m]  of  ℒ
   r = b − Ax
   y = x + V(W^T AV)^{-1} W^T r
end
```

Roughly speeking....

- Accuracy depends on choice of $\mathcal{K}$ and $\mathcal{L}$
- Speed depends on choice of V and W

Projection Classes

Goal: Project x onto $\mathcal{K}$ orthogonal to $\mathcal{L}$

Orthogonal

Assume A is (s)p.d.. and

$$\mathcal{L} = \mathcal{K}$$

A-Orthogonal, oblique

Assume

$$\mathcal{L} = A\mathcal{K}$$

orthog

Theorem

Assume A s.p.d. and $\mathcal{L} = \mathcal{K}$.

Then x_1 is formed from an orthogonal projection onto $\mathcal{K}$ from x_0

iff

x_1 minimizes the A-norm of the error over $x_0 + \mathcal{K}$

So

$$x_1 = \underset{x \in x_0 + \mathcal{K}}{\operatorname{argmin}} \|x^* - x\|_A$$

A-orthog

Theorem

Assume A square and $\mathcal{L} = A\mathcal{K}$.

Then x_1 is formed from a A-orthogonal projection onto $\mathcal{K}$ from x_0

iff

x_1 minimizes the 2-norm of the residual over $x_0 + \mathcal{K}$

So

$$x_1 = \underset{x \in x_0 + \mathcal{K}}{\operatorname{argmin}} \|b - Ax\|_2$$

Goal

Theorem

Assume A square and $\mathcal{L} = A\mathcal{K}$.

Then x_1 is formed from a A-orthogonal projection onto $\mathcal{K}$ from x_0

iff

x_1 minimizes the 2-norm of the residual over $x_0 + \mathcal{K}$

So

$$x_1 = \underset{x \in x_0 + \mathcal{K}}{\operatorname{argmin}} \|b - Ax\|_2$$

Coming up...

- How do we pick $\mathcal{K}$?
- Next time: $\mathcal{K}$ one dimensional.
- Next next time: $\mathcal{K}$ is a Krylov space.