Classical Matrix Splittings: SSOR

Matrix Splitting

$$
\begin{aligned}
(D-\omega L)\underline{x}_{i+\frac{1}{2}} &= ((1-\omega)D+\omega U)\underline{x}_i + \omega\underline{b} \\
(D-\omega U)\underline{x}_{i+1} &= ((1-\omega)D+\omega L)\underline{x}_{i+\frac{1}{2}} + \omega\underline{b}
\end{aligned}
$$

Iteration and Error Equation

$$
\begin{aligned}
\underline{x}_{i+1} &= (D-\omega U)^{-1}((1-\omega)D+\omega L)(D-\omega L)^{-1}((1-\omega)D+\omega U)\underline{x}_i \\
&\quad +\omega(2-\omega)(D-\omega U)^{-1}D(D-\omega L)^{-1}\underline{b} \\
\underline{e}_{i+1} &= (D-\omega U)^{-1}((1-\omega)D+\omega L)(D-\omega L)^{-1}((1-\omega)D+\omega U)\underline{e}_i
\end{aligned}
$$

SSOR Operator

$$
\mathcal{G}_\omega = (D-\omega U)^{-1}((1-\omega)D+\omega L)(D-\omega L)^{-1}((1-\omega)D+\omega U)
$$

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Convergence

- A Irreducibly Diagonally Dominant

$$\mathcal{S}(\mathcal{G}_\omega) < 1 \quad \text{for} \quad 0 < \omega \leq 1$$

- A Hermitian Positive Definite

$$\mathcal{S}(\mathcal{G}_\omega) < 1 \quad \text{for} \quad 0 < \omega < 2$$

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Simple Identity

$$
\begin{aligned}
&((1-\omega)D+\omega L)(D-\omega L)^{-1} \\
&\qquad = D((1-\omega)I+\omega D^{-1}L)(I-\omega D^{-1}L)^{-1}D^{-1} \\
&\qquad = D(I-\omega D^{-1}L)^{-1}((1-\omega)I+\omega D^{-1}L)D^{-1} \\
&\qquad = D(D-\omega L)^{-1}((1-\omega)D+\omega L)D^{-1}
\end{aligned}
$$

Alternate Form of the Iteration

$$
\begin{aligned}
&(D-\omega L)D^{-1}(D-\omega U)\underline{x}_{i+1} \\
&\qquad = ((1-\omega)D+\omega L)D^{-1}((1-\omega)D+\omega U)\underline{x}_i+\omega(2-\omega)\underline{b}
\end{aligned}
$$

Preconditioned One-step Iteration

$$
\begin{aligned}
(D-\omega L)D^{-1}(D-\omega U)\underline{x}_{i+1} &= (D-\omega L)D^{-1}(D-\omega U)\underline{x}_i \\
&+ \omega(2-\omega)(\underline{b}-(D-(L+U))\underline{x}_i)
\end{aligned}
$$

$$
M\underline{x}_{i+1} = M\underline{x}_i + \alpha\underline{r}_i
$$

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Preconditioning $\quad M^{-1}A\underline{x} = M^{-1}\underline{b}$

$$M = \frac{1}{\omega(2-\omega)}(D-\omega L)D^{-1}(D-\omega U)$$

Stencil of M for Model Problem

$$\left[\frac{1}{2-\omega}\right]\begin{bmatrix} \frac{\omega}{4} & -1 & \\ -1 & \frac{4}{\omega}+\frac{\omega}{2} & -1 \\ & -1 & \frac{\omega}{4} \end{bmatrix}$$

- Invariant of scaling $(\frac{1}{(2-\omega)})$
- Convergence factor depends on $\frac{\lambda_{\max}(M^{-1}A)}{\lambda_{\min}(M^{-1}A)}$

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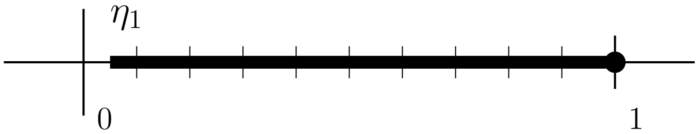
Spectrum for Model Problem

$$\omega_{\text{opt}} = \frac{2}{1 + \sqrt{2(1-\mu_{11})}}$$

$$\Sigma(M^{-1}A) = \Sigma(I - \mathcal{G}_{\omega_{\text{opt}}})$$



$$\eta_1 \;\geq\; \frac{2\sin(\frac{\pi}{2(n+1)})}{1+\sin(\frac{\pi}{2(n+1)})} \;\cong\; \frac{\pi}{n+1}$$

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Convergence for Model Problem

One-step ($\alpha = 1$)

$$\rho \;=\; \mathcal{S}(\mathcal{G}_{\omega_{\rm opt}}) \leq \frac{1 - \sqrt{\frac{1}{2}(1-\mu_1)}}{1 + \sqrt{\frac{1}{2}(1-\mu_1)}} \cong 1 - \frac{\pi}{n+1}$$

$$\varepsilon \;=\; \rho^K \;\Rightarrow\; K \;\cong\; \log(\frac{1}{\varepsilon})\frac{1}{\pi}n$$

One-step ($\alpha = 2/(1+\eta_1)$)

$$\rho = \frac{1-\eta_1}{1+\eta_1} \;\Rightarrow\; K \cong \log(\frac{1}{\varepsilon})\frac{1}{2\pi}n$$

Chebyshev

$$\rho = \left(\frac{\sqrt{1/\eta_1} - 1}{\sqrt{1/\eta_1} + 1}\right) \;\Rightarrow\; K \cong \log(\frac{1}{\varepsilon})\frac{1}{2\sqrt{\pi}}n^{1/2}$$

Classical Matrix Splittings: Summary

Jacobi: $M = D$

One-step ($\alpha = 1$)	$K \cong \log(\frac{1}{\varepsilon})\frac{2}{\pi^2}n^2$
Chebyshev	$K \cong \log(\frac{1}{\varepsilon})\frac{1}{\pi}n$

Gauss-Seidel: $M = D - L$

One-step ($\alpha = 1$)	$K \cong \log(\frac{1}{\varepsilon})\frac{1}{\pi^2}n^2$
One-step ($\alpha = 2 - O(\frac{1}{n^2})$)	$K \cong \log(\frac{1}{\varepsilon})\frac{1}{2\pi^2}n^2$
Chebyshev	$K \cong \log(\frac{1}{\varepsilon})\frac{1}{2\pi}n$

SOR: $M = \frac{1}{\omega}(D - \omega L); \quad \omega_b \cong 2 - \frac{2\pi}{n+1}$

One-step ($\alpha = 1$)	$K \cong \log(\frac{1}{\varepsilon})\frac{1}{2\pi}n$

SSOR: $M = \frac{1}{\omega(2-\omega)}(D - \omega L)D^{-1}(D - \omega U); \quad \omega_{\text{opt}} \cong (2 - \frac{2\pi}{n+1})$

One-step ($\alpha = 1$)	$K \cong \log(\frac{1}{\varepsilon})\frac{1}{\pi}n$
One-step ($\alpha = 2 - O(\frac{1}{n})$)	$K \cong \log(\frac{1}{\varepsilon})\frac{1}{2\pi}n$
Chebyshev	$K \cong \log(\frac{1}{\varepsilon})\frac{1}{2\sqrt{\pi}}n^{1/2}$