Classical Matrix Splittings: Gauss-Seidel

System

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

Splitting

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = - \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & 0 & 0 & a_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

Iteration

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}^{k} = - \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & 0 & 0 & a_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}^{k-1} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

Classical Matrix Splittings: Gauss-Seidel

Splitting

$$A = M - N$$

$$M = D - L \qquad \text{(Diagonal and Lower Triangular)}$$

$$N = U \qquad \text{(Negative Upper Triangular)}$$

Iteration and Error Equation

$$(D - L)\underline{x}_k = U\underline{x}_{k-1} + \underline{b}$$

$$(D - L)\underline{e}_k = U\underline{e}_{k-1}$$

$$\underline{e}_k = (D - L)^{-1}U\underline{e}_{k-1}$$

Classical Matrix Splittings: Gauss-Seidel

Convergence

- A Irreducibly Diagonally Dominant

$$|a_{ii}| \geq \sum_{j=0}^{N} |a_{ij}| \qquad \forall i \qquad (> \text{for some } i)$$

- A Hermitian Positive Definite

- If $D^{-1}(L+U) \geq 0$, then one of the following holds

$$1. \quad \mathcal{S}((D-L)^{-1}U) < \mathcal{S}(D^{-1}(U+L)) < 1$$

$$2. \quad \mathcal{S}((D-L)^{-1}U) = \mathcal{S}(D^{-1}(U+L)) = 1$$

$$3. \quad \mathcal{S}((D-L)^{-1}U) > \mathcal{S}(D^{-1}(U+L)) > 1$$

- Model Problem

$$\mathcal{S}((D-L)^{-1}U) = \mathcal{S}(D^{-1}(U+L))^2 < 1$$

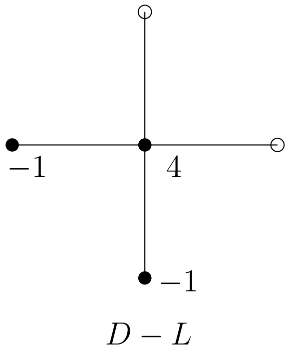
Classical Matrix Splittings: Gauss-Seidel

Stencil of M for Model Problem

$$(D-L)\underline{x}_k = U\underline{x}_{k-1} + \underline{b}$$

$$(D-L)\underline{e}_k = U\underline{e}_{k-1}$$

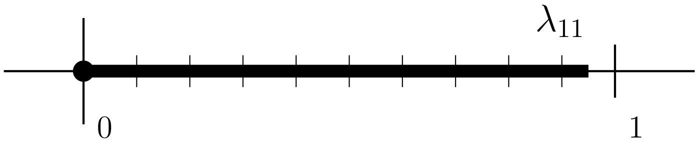
$$\underline{e}_k = (D-L)^{-1}U\underline{e}_{k-1}$$



Spectrum for the Model Problem

If $\lambda_{k\ell} \in \Sigma((D-L)^{-1}U)$ and $\mu_{k\ell} \in \Sigma(D^{-1}(L+U))$, then

$$\lambda_{k\ell} = \mu_{k\ell}^2 = \frac{1}{4}(\cos(\frac{k\pi}{n+1}) + \cos(\frac{\ell\pi}{n+1}))^2$$



$$\lambda_{11} = 1 - \sin^2\left(\frac{\pi}{(n+1)}\right)$$

Classical Matrix Splittings: Gauss-Seidel

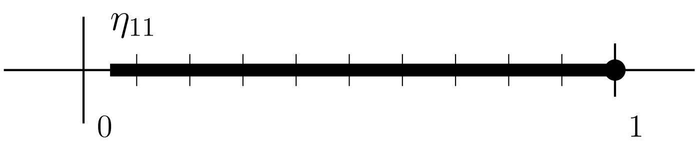
Preconditioning

$$
\begin{aligned}
A\underline{x} &= \underline{b} \\
M &= (D - L) \\
(D-L)^{-1}A\underline{x} &= (D-L)^{-1}\underline{b}
\end{aligned}
$$

Spectrum of Preconditioned System

$$\Sigma(M^{-1}A) = \Sigma(I - (D-L)^{-1}U)$$

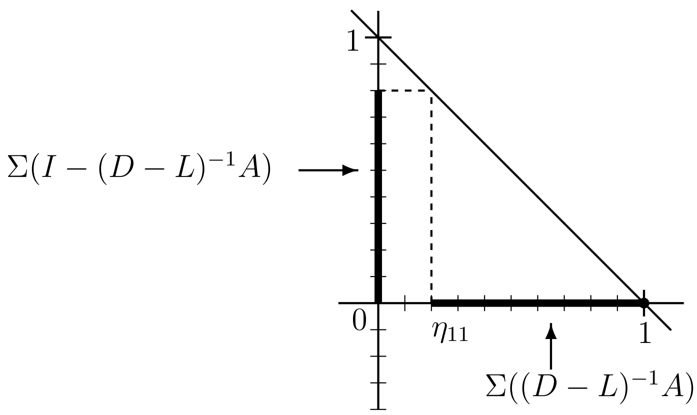
$$\eta_{k\ell} = 1 - \lambda_{k\ell} = 1 - \frac{1}{4}(\cos(\frac{k\pi}{n+1}) + \cos(\frac{\ell\pi}{n+1}))^2$$



$$\eta_{11} \;=\; 1 - (\cos(\frac{\pi}{n+1}))^2 \;\cong\; \sin^2(\frac{\pi}{n+1})$$

Classical Matrix Splittings: Gauss-Seidel

Stationary One-step Method $\alpha = 1$



$$\rho \;=\; \mathcal{S}(I - M^{-1}A) = 1 - \sin^2(\frac{\pi}{n+1})$$

$$\varepsilon \;=\; \rho^K \;\Rightarrow\; K \;\cong\; \log(\frac{1}{\varepsilon})\frac{1}{\pi^2}n^2$$

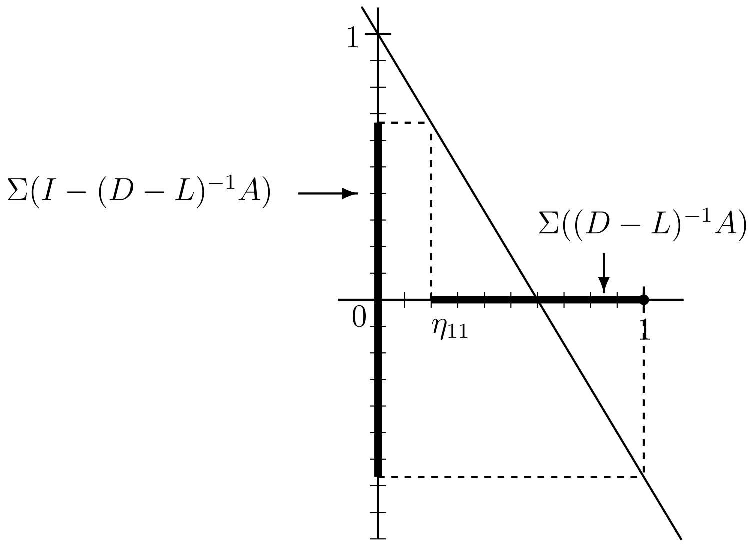
Recall

Jacobi$(1 - \text{step})$ $K \cong \log(\frac{1}{\varepsilon})\frac{2}{\pi^2}n^2$

Jacobi(Chebyshev) $K \cong \log(\frac{1}{\varepsilon})\frac{1}{\pi}n$

Classical Matrix Splittings: Gauss-Seidel

Optimal $\alpha = \frac{2}{1+\eta_{11}} = \frac{2}{1+\sin^2(\frac{\pi}{n+1})}$



$$\begin{aligned} \rho &= \mathcal{S}(I-\alpha(D-L)^{-1}A) \\ &= \left(\frac{1-\eta_{11}}{1+\eta_{11}}\right) = \frac{1-\sin^2(\frac{\pi}{n+1})}{1+\sin^2(\frac{\pi}{n+1})} \end{aligned}$$

$$\varepsilon = \rho^K \;\Rightarrow\; K \;=\; \log(\frac{1}{\varepsilon})\frac{1}{2}(\frac{1}{\eta_{11}}) \;=\; \log\frac{1}{2\pi^2}n^2$$

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Chebyshev Iteration

$$\rho = \left(\frac{\sqrt{1/\eta_{11}} - 1}{\sqrt{1/\eta_{11}} + 1} \right)$$

$$\varepsilon = \rho^K \;\Rightarrow\; K \;\cong\; \log(\frac{1}{\varepsilon})\frac{1}{2}\sqrt{\frac{1}{\eta_{11}}} \;\cong\; \log(\frac{1}{\varepsilon})\frac{1}{2\pi}n$$

Classical Matrix Splittings: SOR

Splitting

$$A = M - N = D - L - U$$

$$M = \tfrac{1}{\omega}(D - \omega L) \qquad\qquad = \tfrac{1}{\omega}D - L$$

$$N = \tfrac{1}{\omega}((1-\omega)D + \omega U) \qquad\qquad = (\tfrac{1}{\omega} - 1)D + U$$

Iteration and Error Equation

$$\begin{aligned}
\frac{1}{\omega}(D - \omega L)\underline{x}_k &= \frac{1}{\omega}((1-\omega)D + \omega U)\underline{x}_{k-1} + \underline{b} \\
\frac{1}{\omega}(D - \omega L)\underline{e}_k &= \frac{1}{\omega}((1-\omega)D + \omega U)\underline{e}_{k-1} \\
\underline{e}_k &= (D - \omega L)^{-1}((1-\omega)D + \omega U)\underline{e}_{k-1}
\end{aligned}$$

Iteration Matrix

$$\mathcal{L}_\omega = (D - \omega L)^{-1}((1-\omega)D + \omega U)$$

Classical Matrix Splittings: SOR

Convergence

- General Matrices

$$\mathcal{S}(\mathcal{L}_\omega) \geq |\omega - 1|$$

- A Irreducibly Diagonally Dominant

$$\mathcal{S}(\mathcal{L}_\omega) < 1 \quad \text{for} \quad 0 < \omega \leq 1$$

- A Hermitian Positive Definite

$$\mathcal{S}(\mathcal{L}_\omega) < 1 \quad \text{for} \quad 0 < \omega < 2$$

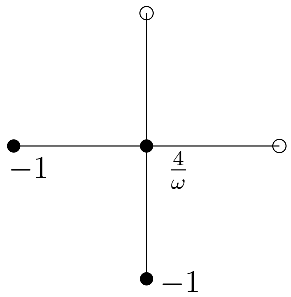
Classical Matrix Splittings: SOR

Stencil of M for Model Problem

$$A = M - N = D - L - U$$

$$M = \frac{1}{\omega}(D - \omega L)$$

$$N = \frac{1}{\omega}((1-\omega)D + \omega U)$$



$\frac{1}{\omega}(D - \omega L)$

Spectrum for the Model Problem

If $\lambda_{k\ell} \in \Sigma(\mathcal{L}_\omega)$ and $\mu_{k\ell} \in \Sigma(D^{-1}(L+U))$, then

$$\lambda_{k\ell}^2 - (\omega^2 \mu_{k\ell}^2 - 2(\omega-1)\lambda_{k\ell} + (\omega-1)^2 = 0$$

Classical Matrix Splittings: SOR

Optimal ω

$$\omega_b \;=\; \frac{2}{1+\sqrt{1-\mu_{11}^2}} = \frac{2}{1+\sin(\frac{\pi}{n+1})}$$

$$\rho_b \;=\; \omega_b - 1 = \frac{1-\sin(\frac{\pi}{n+1})}{1+\sin(\frac{\pi}{n+1})}$$

$$\Sigma(\boldsymbol{\mathcal{L}}_{\omega_b})$$

$$\varepsilon = \rho_b^K \;\Rightarrow\; K \;\cong\; \log(\frac{1}{\varepsilon})\frac{1}{2\pi}n$$

Classical Matrix Splittings: SOR

Preconditioning

$$A\underline{x} = \underline{b} \qquad M^{-1}A\underline{x} = M^{-1}\underline{b}$$

$$M = \frac{1}{\omega}(D - \omega L)$$

$$\Sigma(M^{-1}A)$$

- Optimal ellipse is circle centered at 1.0
- Optimal Chebyshev iteration is 1-step with $\alpha = 1.0$
- No acceleration possible