Preconditioning: Outline

A. Preconditioning/Matrix Splitting

B. Model Problem

C. Classical Splittings

1. Jacobi
2. Gauss-Seidel
3. SOR
4. SSOR

D. Incomplete Factorization

1. IC (Incomplete Cholesky)
2. MIC (Modified Incomplete Cholesky)

E. Equivalent Operators

Preconditioning

Given the system

$$A\underline{x} = \underline{b}$$

A preconditioning is any nonsingular linear process C such that the equivalent system

$$CA\underline{x} = C\underline{b}$$

is in some sense easier to solve.

Preconditioning/Matrix Splitting

Given

$$A\underline{x} = \underline{b}$$

Matrix splitting

$$A = M - N$$

Write

$$M\underline{x} = N\underline{x} + \underline{b}$$

$$M\underline{x}_k = N\underline{x}_{k-1} + \underline{b}$$

Error equation

$$M\underline{e}_k = N\underline{e}_{k-1}$$

$$\underline{e}_k = M^{-1}N\underline{e}_{k-1}$$

Preconditioning/Matrix Splitting

Reformulate Matrix Splitting

$$
\begin{aligned}
M\underline{x}_k &= N\underline{x}_{k-1} + \underline{b} \\
M\underline{x}_k &= M\underline{x}_{k-1} + (\underline{b} - (M - N)\underline{x}_{k-1}) \\
M\underline{x}_k &= M\underline{x}_{k-1} + \underline{r}_{k-1} \\
\underline{x}_k &= \underline{x}_{k-1} + M^{-1}\underline{r}_{k-1}
\end{aligned}
$$

Stationary One-step Method

$$
M^{-1}A\underline{x} = M^{-1}\underline{b}
$$

$$
\underline{x}_k = \underline{x}_{k-1} + \alpha M^{-1}\underline{r}_{k-1}
$$

Matrix splitting is equivalent to the simplest stationary one-step method applied to the original system preconditioned by M^{-1}.

Model Problem

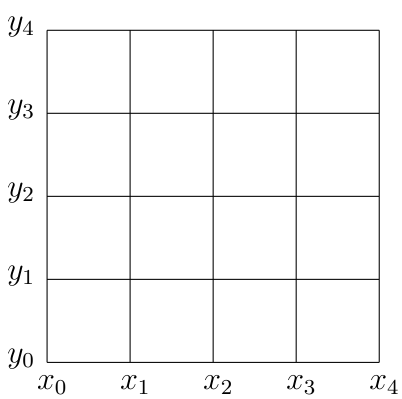
$$-(u_{xx} + u_{yy}) = f \qquad (x, y) \in [0, 1] \times [0, 1]$$

$$\begin{aligned} u(x, 0) &= u(x, 1) = 0 \\ u(0, y) &= u(1, y) = 0 \end{aligned}$$

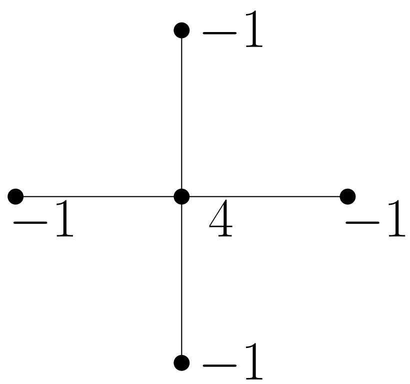
Centered Difference Formula

$$\begin{aligned} \frac{1}{h^2}(-u(x-h, y) + 2u(x, y) - u(x+h, y)) &= -u_{xx}(x, y) + O(h^2) \\ \frac{1}{h^2}(-u(x, y-h) + 2u(x, y) - u(x, y+h)) &= -u_{yy}(x, y) + O(h^2) \end{aligned}$$

Mesh



Stencil



Model Problem

Matrix Problem: Centered Difference Equations

$$u_{ij} \cong u(x_i, y_j)$$

$$f_{ij} = f(x_i, y_j)$$

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & & & \\ -1 & 4 & -1 & 0 & -1 & 0 & & & \\ 0 & -1 & 4 & 0 & 0 & -1 & & & \\ & & & & & & & & \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ & & & & & & & & \\ & & & -1 & 0 & 0 & 4 & -1 & 0 \\ & & & 0 & -1 & 0 & -1 & 4 & -1 \\ & & & 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{21} \\ u_{22} \\ u_{23} \\ u_{31} \\ u_{32} \\ u_{33} \end{bmatrix} = h^2 \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{bmatrix}$$

$$A\underline{u} = \underline{f}$$

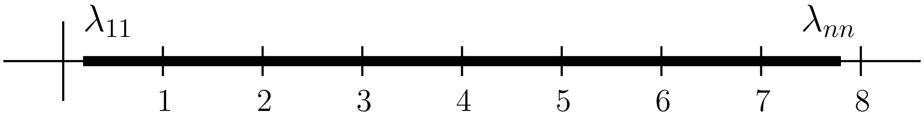
Model Problem

Eigenvector Decomposition $\qquad (h = \frac{1}{n+1})$

$$A\underline{v}_{k\ell} = \lambda_{k\ell}\underline{v}_{k\ell} \qquad k, \ell = 1, \cdots, n \qquad (N = n^2)$$

$$\begin{aligned}\lambda_{k\ell} &= (2 - 2\cos(\frac{k\pi}{n+1})) + (2 - 2\cos(\frac{\ell\pi}{n+1})) \\ &= 4(\sin^2(\frac{k\pi}{2(n+1)}) + \sin^2(\frac{\ell\pi}{2(n+1)}))\end{aligned}$$

$$(\underline{v}_{k\ell})_{ij} = \sin(\frac{k\pi i}{n+1})\sin(\frac{\ell\pi j}{n+1})$$



$$\lambda_{11} = 8\sin^2\left(\frac{\pi}{2(n+1)}\right) \qquad\qquad \lambda_{nn} = 8 - \lambda_{11}$$

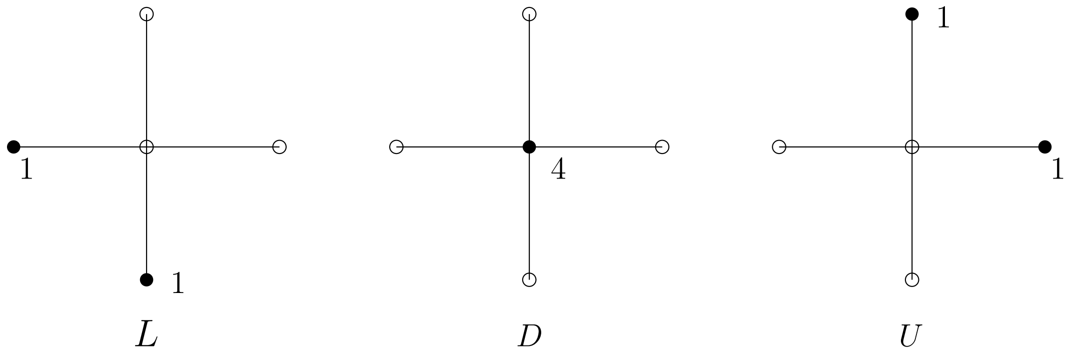
Model Problem

Write

$$A = D - L - U$$

D	Diagonal
L	Lower Triangular
U	Upper Triangular

Stencil



Classical Matrix Splittings: Jacobi

System

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

Splitting

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = - \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

Iteration

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}^{k} = - \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}^{k-1} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

Classical Matrix Splittings: Jacobi

Splitting

$$A = M - N$$

$$M = D \qquad \text{(Diagonal)}$$

$$N = L + U \qquad \text{(Negative Off Diagonal)}$$

Iteration and Error Equation

$$D\underline{x}_k = (L+U)\underline{x}_{k-1} + \underline{b}$$

$$D\underline{e}_k = (L+U)\underline{e}_{k-1}$$

$$\underline{e}_k = D^{-1}(L+U)\underline{e}_{k-1}$$

Convergence

- $\mathcal{S}(D^{-1}(L+U)) < 1$ (Spectral Radius)
- A Irreducibly Diagonally Dominant

$$|a_{ii}| \geq \sum_{j=0}^{N} |a_{ij}| \qquad \forall i \qquad (> \text{ for some } i)$$

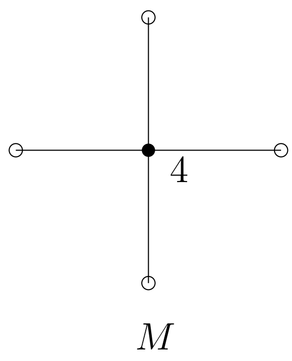
Classical Matrix Splittings: Jacobi

Stencil of M for Model Problem

$$D\underline{x}_k = (L+U)\underline{x}_{k-1} + \underline{b}$$

$$D\underline{e}_k = (L+U)\underline{e}_{k-1}$$

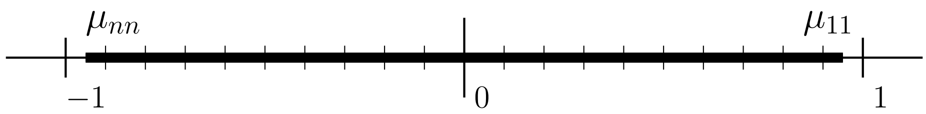
$$\underline{e}_k = D^{-1}(L+U)\underline{e}_{k-1}$$



Spectrum for the Model Problem

If $\mu_{k\ell} \in \Sigma(D^{-1}(L+U))$, then

$$\mu_{k\ell} = \frac{1}{2}(\cos(\frac{k\pi}{n+1}) + \cos(\frac{\ell\pi}{n+1}))$$



$$\mu_{nn} = -\mu_{11} \qquad \mu_{11} = 1 - 2\sin^2\left(\frac{\pi}{2(n+1)}\right)$$

Classical Matrix Splittings: Jacobi

Preconditioning

$$A\underline{x} = \underline{b}$$

$$M = D$$

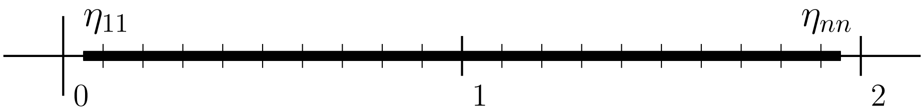
$$D^{-1}A\underline{x} = D^{-1}\underline{b}$$

Spectrum of Preconditioned System

$$\Sigma(D^{-1}A) = \Sigma(I - D^{-1}(L+U))$$

$$D^{-1}A\underline{v}_{k\ell} = \eta_{k\ell}\underline{v}_{k\ell}$$

$$\eta_{k\ell} = 1 - \frac{1}{2}(\cos(\frac{k\pi}{n+1}) + \cos(\frac{\ell\pi}{n+1}))$$



$$\eta_{11} = 2\sin^2\left(\frac{\pi}{2(n+1)}\right) \qquad \mu_{nn} = 2 - \mu_{11}$$

Classical Matrix Splittings: Jacobi

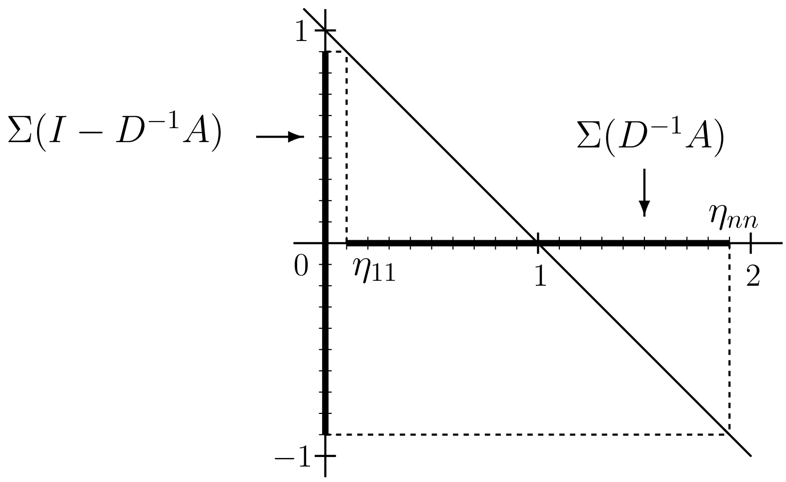
Stationary One-step Method

$$\underline{x}_k = \underline{x}_{k-1} + \alpha D^{-1}\underline{r}_{k-1}$$

$$\underline{e}_k = (I - \alpha D^{-1}A)\underline{e}_{k-1}$$

$$= (I - \alpha(I - D^{-1}(L+U)))\underline{e}_{k-1}$$

Optimum $\alpha = 1.0$

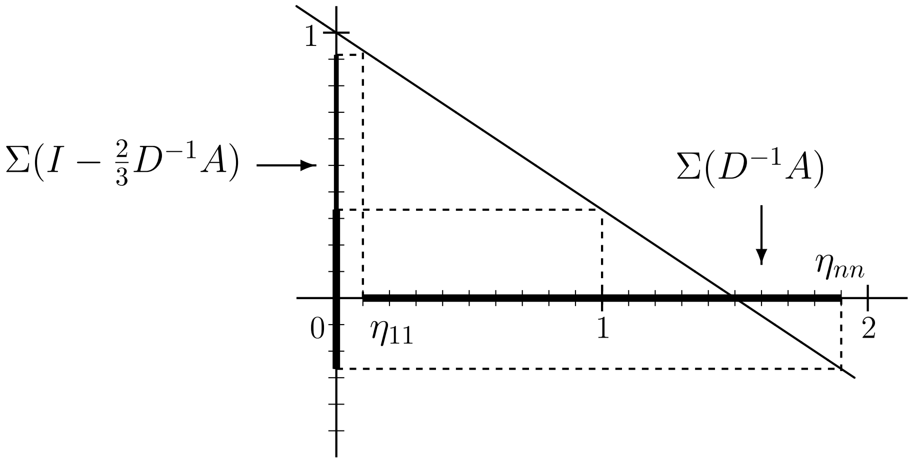


$$\rho = \frac{\eta_{nn} - \eta_{11}}{\eta_{nn} + \eta_{11}} = 1 - 2\sin^2(\frac{\pi}{2(n+1)})$$

$$\varepsilon = \rho^K \quad \Rightarrow \quad K \cong \log(\frac{1}{\varepsilon})\frac{1}{2}\left(\frac{\eta_{nn}}{\eta_{11}}\right) \cong \log(\frac{1}{\varepsilon})\left(\frac{2}{\pi^2}\right)n^2$$

Classical Matrix Splittings: Jacobi

Dampened Jacobi $\quad \alpha = \frac{2}{3}$



$$\mathcal{S}(I - \frac{2}{3}D^{-1}A) = 1 - \frac{4}{3}\sin^2\left(\frac{\pi}{2(n+1)}\right)$$

$$\mathcal{S}(\text{HighFrequencies}) = \frac{1}{3}$$

Classical Matrix Splittings: Jacobi

Chebyshev Iteration (or CG)

$$
\begin{aligned}
\underline{x}_k &= \underline{x}_{k-1} + \underline{\delta}_{k-1} \\
\underline{\delta}_k &= \alpha_k \underline{r}_k + \beta_k \underline{\delta}_{k-1} \\
\underline{e}_k &= p_k(D^{-1}A)\underline{e}_0
\end{aligned}
$$

$$
\rho = \left(\frac{\sqrt{\eta_{nn}/n_{11}} - 1}{\sqrt{\eta_{nn}/\eta_{11}} + 1} \right)
$$

$$
\varepsilon = \rho^K \quad \Rightarrow \quad K \cong \log(\frac{1}{\varepsilon})\frac{1}{2}\sqrt{\frac{\eta_{nn}}{\eta_{11}}} \cong \log(\frac{1}{\varepsilon})\frac{1}{\pi}n
$$