

*A process cannot be understood by stopping it. Understanding must move with the flow of the process, must join it and flow with it.*

— The First Law of Mentat, in Frank Herbert's *Dune* (1965)

*There's a difference between knowing the path and walking the path.*

— Morpheus [Laurence Fishburne], *The Matrix* (1999)

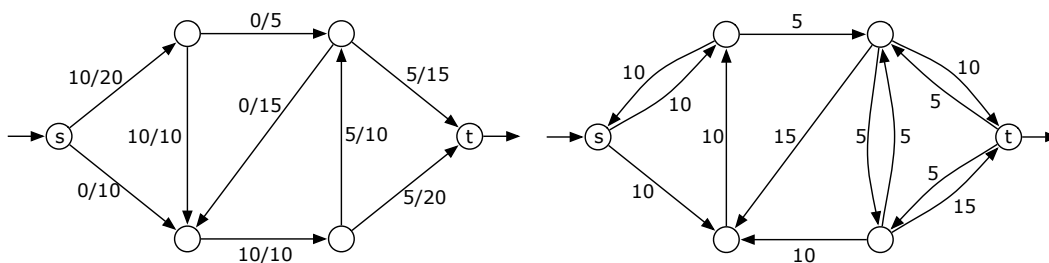
## 13 Max-Flow Algorithms

### 13.1 Recap

Fix a directed graph  $G = (V, E)$  that does not contain both an edge  $u \rightarrow v$  and its reversal  $v \rightarrow u$ , and fix a capacity function  $c : E \rightarrow \mathbb{R}_{\geq 0}$ . For any flow function  $f : E \rightarrow \mathbb{R}_{\geq 0}$ , the *residual capacity* is defined as

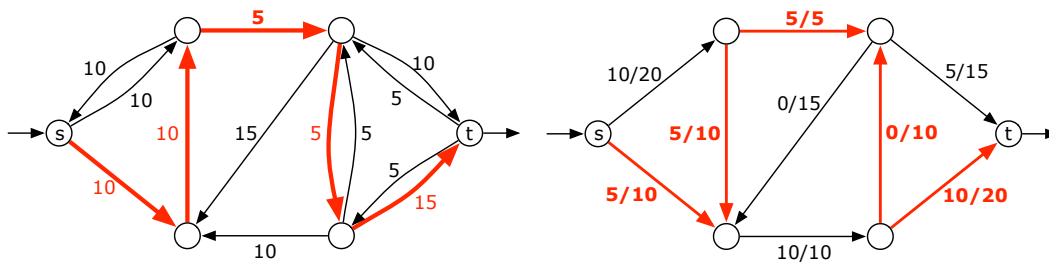
$$c_f(u \rightarrow v) = \begin{cases} c(u \rightarrow v) - f(u \rightarrow v) & \text{if } u \rightarrow v \in E \\ f(v \rightarrow u) & \text{if } v \rightarrow u \in E \\ 0 & \text{otherwise} \end{cases}$$

The *residual graph*  $G_f = (V, E_f)$ , where  $E_f$  is the set of edges whose non-zero residual capacity is positive.



A flow  $f$  in a weighted graph  $G$  and its residual graph  $G_f$ .

In the last lecture, we proved the Max-flow Min-cut Theorem: *In any weighted directed graph network, the value of the maximum  $(s, t)$ -flow is equal to the cost of the minimum  $(s, t)$ -cut.* The proof of the theorem is constructive. If the residual graph contains a path from  $s$  to  $t$ , then we can increase the flow by the minimum capacity of the edges on this path, so we must not have the maximum flow. Otherwise, we can define a cut  $(S, T)$  whose cost is the same as the flow  $f$ , such that every edge from  $S$  to  $T$  is saturated and every edge from  $T$  to  $S$  is empty, which implies that  $f$  is a maximum flow and  $(S, T)$  is a minimum cut.



An augmenting path in  $G_f$  and the resulting (maximum) flow  $f'$ .

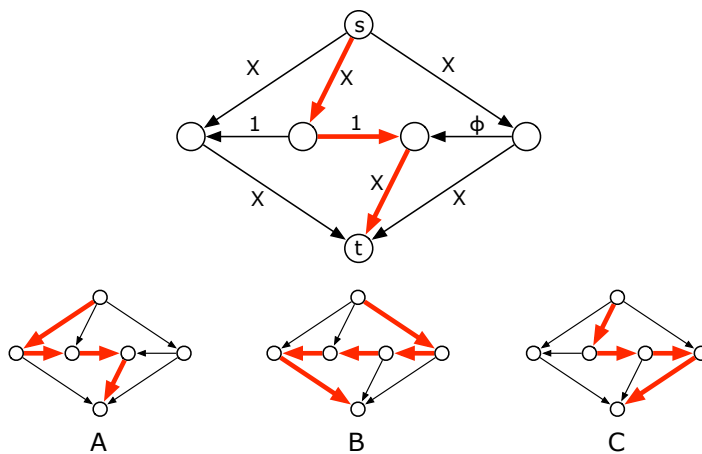
### 13.2 Ford-Fulkerson

It's not hard to realize that this proof translates almost immediately to an algorithm, first developed by Ford and Fulkerson in the 1950s: Starting with the zero flow, repeatedly augment the flow along **any** path  $s \rightsquigarrow t$  in the residual graph, until there is no such path.

If every edge capacity is an integer, then every augmentation step increases the value of the flow by a positive integer. Thus, the algorithm halts after  $|f^*|$  iterations, where  $f^*$  is the actual maximum flow. Each iteration requires  $O(E)$  time, to create the residual graph  $G_f$  and perform a whatever-first-search to find an augmenting path. Thus, in the worst case, the Ford-Fulkerson algorithm runs in  $O(E|f^*|)$  time.

If we multiply all the capacities by the same (positive) constant, the maximum flow increases everywhere by the same constant factor. It follows that if all the edge capacities are *rational*, then the Ford-Fulkerson algorithm eventually halts. However, if we allow irrational capacities, the algorithm can loop forever, always finding smaller and smaller augmenting paths. Worse yet, this infinite sequence of augmentations may not even converge to the maximum flow! Perhaps the simplest example of this effect was discovered by Uri Zwick.

Consider the graph shown below, with six vertices and nine edges. Six of the edges have some large integer capacity  $X$ , two have capacity 1, and one has capacity  $\phi = (\sqrt{5} - 1)/2 \approx 0.618034$ , chosen so that  $1 - \phi = \phi^2$ . To prove that the Ford-Fulkerson algorithm can get stuck, we can watch the residual capacities of the three horizontal edges as the algorithm progresses. (The residual capacities of the other six edges will always be at least  $X - 3$ .)



Uri Zwick's non-terminating flow example, and three augmenting paths.

The Ford-Fulkerson algorithm starts by choosing the central augmenting path, shown in the large figure above. The three horizontal edges, in order from left to right, now have residual capacities 1, 0,  $\phi$ . Suppose inductively that the horizontal residual capacities are  $\phi^{k-1}$ , 0,  $\phi^k$  for some non-negative integer  $k$ .

1. Augment along  $B$ , adding  $\phi^k$  to the flow; the residual capacities are now  $\phi^{k+1}$ ,  $\phi^k$ , 0.
2. Augment along  $C$ , adding  $\phi^k$  to the flow; the residual capacities are now  $\phi^{k+1}$ , 0,  $\phi^k$ .
3. Augment along  $B$ , adding  $\phi^{k+1}$  to the flow; the residual capacities are now 0,  $\phi^{k+1}$ ,  $\phi^{k+2}$ .
4. Augment along  $A$ , adding  $\phi^{k+1}$  to the flow; the residual capacities are now  $\phi^{k+1}$ , 0,  $\phi^{k+2}$ .

Thus, after  $4n + 1$  augmentation steps, the residual capacities are  $\phi^{2n-2}, 0, \phi^{2n-1}$ . As the number of augmentation steps grows to infinity, the value of the flow converges to

$$1 + 2 \sum_{i=1}^{\infty} \phi^i = 1 + \frac{2}{1 - \phi} = 4 + \sqrt{5} < 7,$$

even though the maximum flow value is clearly  $2X + 1$ .

Picky students might wonder at this point why we care about irrational capacities; after all, computers can't represent anything but (small) integers or (dyadic) rationals exactly. Good question! One reason is that the integer restriction is literally *artificial*; it's an *artifact* of actual computational hardware<sup>1</sup>, not an inherent feature of the abstract mathematical problem. Another reason, which is probably more convincing to most practical computer scientists, is that the behavior of the algorithm with irrational inputs tells us something about its worst-case behavior *in practice* given floating-point capacities—terrible! Even with very reasonable capacities, a careless implementation of Ford-Fulkerson could enter an infinite loop simply because of round-off error!

### 13.3 Edmonds-Karp: Fat Pipes

The Ford-Fulkerson algorithm does not specify which alternating path to use if there is more than one. In 1972, Jack Edmonds and Richard Karp analyzed two natural heuristics for choosing the path. The first is essentially a greedy algorithm:

Choose the augmenting path with largest bottleneck value.

It's a fairly easy to show that the maximum-bottleneck  $(s, t)$ -path in a directed graph can be computed in  $O(E \log V)$  time using a variant of Jarník's minimum-spanning-tree algorithm, or of Dijkstra's shortest path algorithm. Simply grow a directed spanning tree  $T$ , rooted at  $s$ . Repeatedly find the highest-capacity edge leaving  $T$  and add it to  $T$ , until  $T$  contains a path from  $s$  to  $t$ . Alternately, one could emulate Kruskal's algorithm—insert edges one at a time in decreasing capacity order until there is a path from  $s$  to  $t$ —although this is less efficient.

We can now analyze the algorithm in terms of the value of the maximum flow  $f^*$ . Let  $f$  be any flow in  $G$ , and let  $f'$  be the maximum flow in the current residual graph  $G_f$ . (At the beginning of the algorithm,  $G_f = G$  and  $f' = f^*$ .) Let  $e$  be the bottleneck edge in the next augmenting path. Let  $S$  be the set of vertices reachable from  $s$  through edges with capacity greater than  $c(e)$  and let  $T = V \setminus S$ . By construction,  $T$  is non-empty, and every edge from  $S$  to  $T$  has capacity at most  $c(e)$ . Thus, the cost of the cut  $(S, T)$  is at most  $c(e) \cdot E$ . On the other hand,  $\|S, T\| \geq |f|$ , which implies that  $c(e) \geq |f|/E$ .

Thus, augmenting  $f$  along the maximum-bottleneck path in  $G_f$  multiplies the maximum flow value in  $G_f$  by a factor of at most  $1 - 1/E$ . In other words, the residual flow *decays exponentially* with the number of iterations. After  $E \cdot \ln |f^*|$  iterations, the maximum flow value in  $G_f$  is at most

$$|f^*| \cdot (1 - 1/E)^{E \cdot \ln |f^*|} < |f^*| e^{-\ln |f^*|} = 1.$$

(That's Euler's constant  $e$ , not the edge  $e$ . Sorry.) In particular, *if all the capacities are integers*, then after  $E \cdot \ln |f^*|$  iterations, the maximum capacity of the residual graph is zero and  $f$  is a maximum flow.

We conclude that for graphs with integer capacities, the Edmonds-Karp 'fat pipe' algorithm runs in  $O(E^2 \log E \log |f^*|)$  time.

<sup>1</sup>...or perhaps the laws of physics. Yeah, right. Whatever. Like *reality* actually matters in this class.

### 13.4 Dinits/Edmonds-Karp: Short Pipes

The second Edmonds-Karp heuristic was actually proposed by Ford and Fulkerson in their original max-flow paper, and first analyzed by the Russian mathematician Dinits (sometimes transliterated Dinic) in 1970. Edmonds and Karp published their independent and slightly weaker analysis in 1972. So naturally, almost everyone refers to this algorithm as ‘Edmonds-Karp’.<sup>2</sup>

Choose the augmenting path with fewest edges.

The correct path can be found in  $O(E)$  time by running breadth-first search in the residual graph. More surprisingly, the algorithm halts after a polynomial number of iterations, independent of the actual edge capacities!

The proof of this upper bound relies on two observations about the evolution of the residual graph. Let  $f_i$  be the current flow after  $i$  augmentation steps, let  $G_i$  be the corresponding residual graph. In particular,  $f_0$  is zero everywhere and  $G_0 = G$ . For each vertex  $v$ , let  $level_i(v)$  denote the unweighted shortest path distance from  $s$  to  $v$  in  $G_i$ , or equivalently, the *level* of  $v$  in a breadth-first search tree of  $G_i$  rooted at  $s$ .

Our first observation is that these levels can only increase over time.

**Lemma 1.**  $level_{i+1}(v) \geq level_i(v)$  for all vertices  $v$  and integers  $i$ .

**Proof:** The claim is trivial for  $v = s$ , since  $level_i(s) = 0$  for all  $i$ . Choose an arbitrary vertex  $v \neq s$ , and let  $s \rightarrow \dots \rightarrow u \rightarrow v$  be a shortest path from  $s$  to  $v$  in  $G_{i+1}$ . (If there is no such path, then  $level_{i+1}(v) = \infty$ , and we’re done.) Because this is a shortest path, we have  $level_{i+1}(v) = level_{i+1}(u) + 1$ , and the inductive hypothesis implies that  $level_{i+1}(u) \geq level_i(u)$ .

We now have two cases to consider. If  $u \rightarrow v$  is an edge in  $G_i$ , then  $level_i(v) \leq level_i(u) + 1$ , because the levels are defined by breadth-first traversal.

On the other hand, if  $u \rightarrow v$  is not an edge in  $G_i$ , then  $v \rightarrow u$  must be an edge in the  $i$ th augmenting path. Thus,  $v \rightarrow u$  must lie on the shortest path from  $s$  to  $t$  in  $G_i$ , which implies that  $level_i(v) = level_i(u) - 1 \leq level_i(u) + 1$ .

In both cases, we have  $level_{i+1}(v) = level_{i+1}(u) + 1 \geq level_i(u) + 1 \geq level_i(v)$ . □

Whenever we augment the flow, the bottleneck edge in the augmenting path disappears from the residual graph, and some other edge in the *reversal* of the augmenting path may (re-)appear. Our second observation is that an edge cannot appear or disappear too many times.

**Lemma 2.** During the execution of the Dinits/Edmonds-Karp algorithm, any edge  $u \rightarrow v$  disappears from the residual graph  $G_f$  at most  $V/2$  times.

**Proof:** Suppose  $u \rightarrow v$  is in two residual graphs  $G_i$  and  $G_{j+1}$ , but not in any of the intermediate residual graphs  $G_{i+1}, \dots, G_j$ , for some  $i < j$ . Then  $u \rightarrow v$  must be in the  $i$ th augmenting path, so  $level_i(v) = level_i(u) + 1$ , and  $v \rightarrow u$  must be on the  $j$ th augmenting path, so  $level_j(v) = level_j(u) - 1$ . By the previous lemma, we have

$$level_j(u) = level_j(v) + 1 \geq level_i(v) + 1 = level_i(u) + 2.$$

<sup>2</sup>To be fair, Edmonds and Karp discovered their algorithm a few years before publication—getting ideas into print takes time, especially in the early 1970s—which is why some authors believe they deserve priority. I don’t buy it; Dinits also presumably discovered his algorithm a few years before its publication. (In Soviet Union, result publish you.) On the gripping hand, Dinits’s paper also described an improvement to the algorithm presented here that runs in  $O(V^2E)$  time instead of  $O(VE^2)$ , so maybe *that* ought to be called Dinits’s algorithm.

In other words, the distance from  $s$  to  $u$  increased by at least 2 between the disappearance and reappearance of  $u \rightarrow v$ . Since every level is either less than  $V$  or infinite, the number of disappearances is at most  $V/2$ .  $\square$

Now we can derive an upper bound on the number of iterations. Since each edge can disappear at most  $V/2$  times, there are at most  $EV/2$  edge disappearances overall. But at least one edge disappears on each iteration, so the algorithm must halt after at most  $EV/2$  iterations. Finally, since each iteration requires  $O(E)$  time, Dinits' algorithm runs in  $O(VE^2)$  time overall.

## Exercises

- ★1. Describe a directed graph with irrational edge capacities, such that the Edmonds-Karp ‘fat pipe’ heuristic does not halt.
2. We can speed up the Edmonds-Karp ‘fat pipe’ heuristic, at least for integer capacities, by relaxing our requirements for the next augmenting path. Instead of finding the augmenting path with maximum bottleneck capacity, we find a path whose bottleneck capacity is at least half of maximum, using the following *scaling* algorithm.

The algorithm maintains a bottleneck threshold  $\Delta$ ; initially,  $\Delta$  is the maximum capacity among all edges in the graph. In each *phase*, the algorithm augments along paths from  $s$  to  $t$  in which every edge has residual capacity at least  $\Delta$ . When there is no such path, the phase ends, we set  $\Delta \leftarrow \Delta/2$ , and the next phase begins.

- (a) How many phases will the algorithm execute in the worst case, if the edge capacities are integers?
- (b) Let  $f$  be the flow at the end of a phase for a particular value of  $\Delta$ . Let  $S$  be the nodes that are reachable from  $s$  in the residual graph  $G_f$  using only edges with residual capacity at least  $\Delta$ , and let  $T = V \setminus S$ . Prove that the cost (with respect to  $G$ 's original edge capacities) of the cut  $(S, T)$  is at most  $|f| + E \cdot \Delta$ .
- (c) Prove that in each phase of the scaling algorithm, there are at most  $2E$  augmentations.
- (d) What is the overall running time of the scaling algorithm, assuming all the edge capacities are integers?