CS 421 – Spring 2007
Lecture Notes Set 27:
Lambda Calculus Continued

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Corresponding to Slides: `10-lambda-intro (slides 18-28)`
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1 Change Log

1.0 Initial Release.

2 α eqivalence (slides 18-19)

Last time we talked about congruency. The we extended that idea to α eqivalence. Here we said that α eqivalence is the smallest congruent term that has an α conversion.

Let's look at an example to better understand this. We want to show that λ x. (λ y. y x) x is α equivalent to λ y. (λ x. x y) y

Our first step is to convert the x's to z's. We can do this with α conversion because allows for conversions when a variable isn't restricted by being bound. So now we have:

λ x. (λ y. y x) x ˜ λ z. (λ y. y z) z

Now for the inner λ, the there are no x's bound in there, so we can convert the y's to x's and get:

λ z. (λ y. y z) z ˜ λ z. (λ x. x z) z

Here, we can replace the z's with y's (just like we converted the x's to z's earlier.

λ z. (λ x. x z) z ˜ λ y. (λ x. x y) y

So now we just showed that we were able to convert λ x. (λ y. y x) x to λ y. (λ x. x y) y.

3 η Reduction (slide 20)

Now, let's take a look at η reduction. Here we say that if we have some equation, λ x. f x, with η reduction, we can simplify this too f. Not that this is not the same as saying (λ x. f) x. So the different here is that x in the second equation is a free variable, while it is bound in the first. To reduce it, it has to be bound. If it were free, that is an instance of β reduction.

An example of η reduction is λ x. (λ y. y) x is equivalent to λ y. y

4 Substitution (slides 21-24)

With α equivalence, there is also the notion of substitution. Where you see [N / x] P, it means for every x that you see, that is free, in the term P, you replace it with N only if you don't end up binding anything new during the substitution. To avoid this, you can rename bound variables in P.

Let's look at a couple examples. If you had [N / x] x you would just get N after the substitution. That is because that line is saying, for every x we see in x, replace it with N. So that is what we did. But if you had [N / x] y, you would get y back (provided that x did not equal y). That is because the line is saying, for every x we see in y, replace it with N. There are no x's in y, so we replace nothing.

So what is then the problem with doing [(λ x. x y) / z] (λ y. y z). The problems with this is that the z is in the scope of y. So replacing the z with something that has a y in it will bind the y. That is, we'll get: λ y. y (λ x. x y).

But we want the inner y to be free. So to fix it, we can do a simple α conversion and turn (λ y. y z) to (λ w. w z). Now, when we replace z, we have λ w. w (λ x. x y)

Note, that you only replace the free variables. So if you had [(λ x. x) / z] (λ y. y z (λ z. z)). That doesn't equal λ y. y (λ x. x) (λ z. (λ x. x)) but rather λ y. y (λ x. x) (λ z. z)

5 β reduction (slides 25-26)

β reduction is also defined by α equivalences and is the essence of computation in λ calculus. The idea is that when you have the an equation with something like (λ x. P) N, you can reduce it too simply saying, [N / x] P or for every x you see in P, replace it with N. This is basically the idea of applying an argument to a function.

Here is an example. If we started (λ z. (λ x. x y) z) (λ y. y z), our first step would be to get rid of the first z by η reduction. So then we would have (λ x. x y)(λ y. y z). The next step would be to do the β reduction and replace the x's. So we have (λ y. y z) y. Then after that, we do another η reduction and end up with simply yz as our final answer.

6 $\alpha\beta$ equivalence (slides 27-28)

This is the smallest congruency that has an α equivalence and β reduction. Also, we say that a term is in normal form if there exists no subterm that is α equivalent to a term that can be β reduced. We also say that if two expressions are $\alpha\beta$ equivalent and both in normal forms, then they are also α equivalent.

Note that not all terms will reduce to normal forms. Also, not all reduction strategies will prodice a normal form if there is one, so you may have to play around with them to see what is going on.