CS 273 4/17/07

More Undecidability!

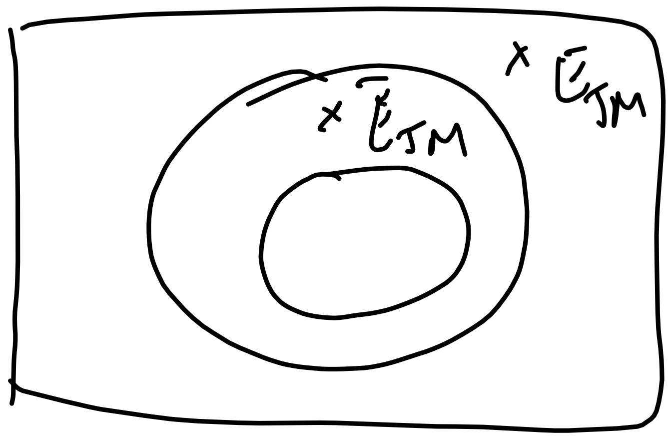
- Reductions
- LBAs
- Computation Histories
- Language Problems.

$E_{TM} = \{ \langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset \}$

is undecidable.

E_{TM} is not Turing Recogn

$\overline{E_{TM}}$ is Turing recognizable



$$\text{Regular}_{TM} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is regular}\}$$

Reduction is from

$$A_{TM} = \{\langle M, w \rangle \mid \text{TM } M \text{ accepts } w\}$$

Auxiliary TM: $\Sigma = \{0,1\}$

$R_{\langle M, w\rangle}$: on input x

1. If x is $0^n 1^n$ (for $n \geq 0$) accept

2. Otherwise simulate M on w

3. If M accepts w then accept x

4. If M halts and rejects w, reject x

$L(R_{\langle M,w\rangle}) = \Sigma^*$ if M accepts w

$= \{0^n 1^n \mid n \geq 0\}$ if M does not accept w.

Reduce A_{TM} to $Regular_{TM}$

Let H be a decider for $Regular_{TM}$. create a decider N for A_{TM}

N: on input $\langle M, w \rangle$

1. create $\langle R_{\langle M,w \rangle} \rangle$
2. Run H on $\langle R_{\langle M,w \rangle} \rangle$
3. If H accepts then accept
4. If H rejects then reject

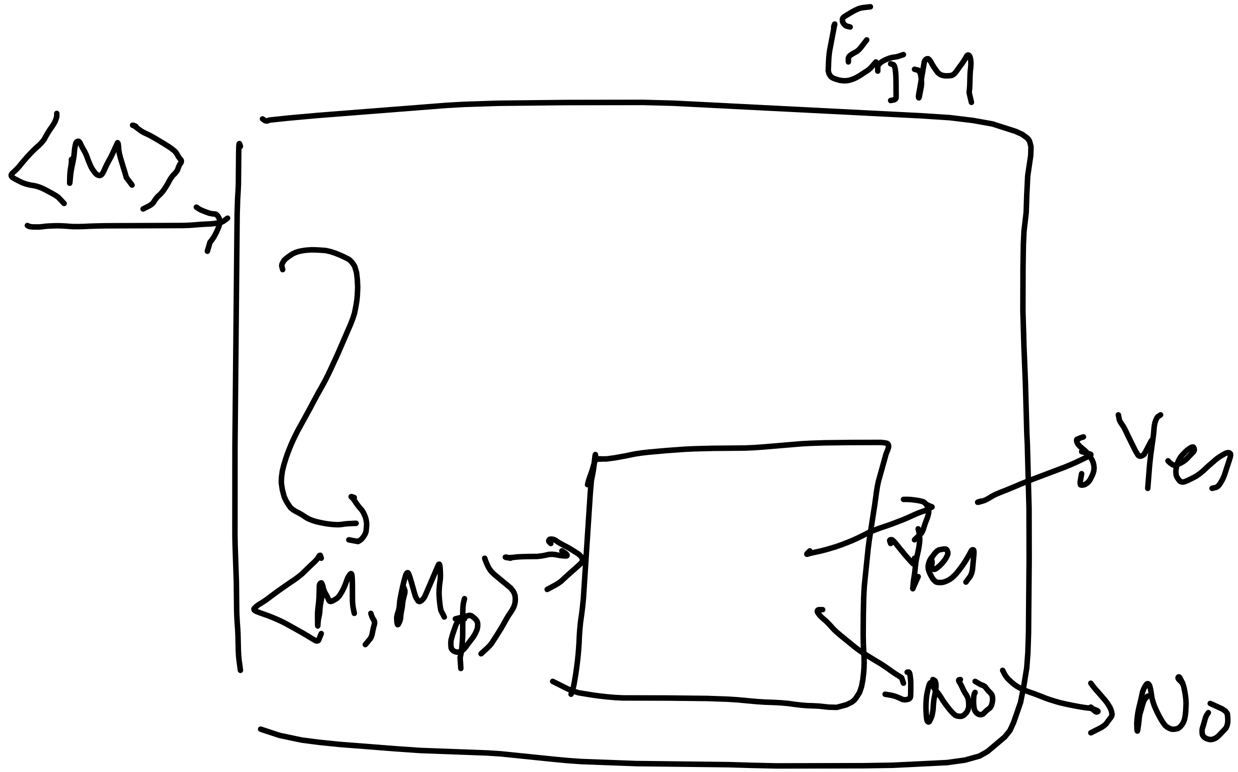
$$EQ_{TM} = \{ \langle M_1, M_2 \rangle \mid M_1, M_2 \text{ TMs and } L(M_1) = L(M_2) \}$$

Reduction from E_{TM}?

$$L = \{ \langle M \rangle \mid M \text{ is a TM } L(M) = \phi \}$$

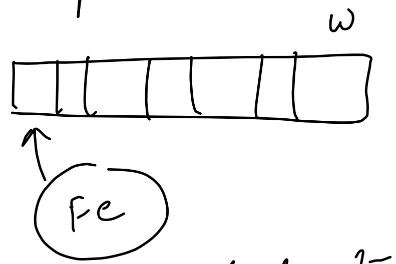
M_ϕ : on input x

1. Reject!

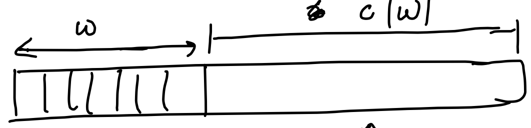


Linear Bounded Automaton

A TM which can only use the space used by the input



This is equivalent to



scratch space

"linear" amount of ↑

Non-deterministic LBA is equivalent to Context sensitive languages

Open problem: are deterministic LBA's equivalent to non-deterministic LBAs.

① $A_{LBA} = \{\langle M, w\rangle \mid M \text{ is a LBA and } M \text{ accepts } w\}$

is decidable.

② $E_{LBA} = \{\langle M\rangle \mid M \text{ is a LBA } L(M) = \phi\}$

is undecidable.

① A_{LBA} is decidable

Basic idea:

Simulate M on w.

What is a configuration of an LBA?

$\alpha q \beta$

Given an LBA with $|Q|$ states and tape alphabet Γ and input w how many distinct configurations are there?

$$|Q| \cdot |w| \cdot |\Gamma|^{|w|}$$

Decider for A_{LBA}

N: on input $\langle M, w \rangle$

1. Simulate M on w for $|Q| \cdot |w| \cdot |\Gamma|^{|w|}$ steps or until M halts.
2. If M halts return M's answer.
3. Otherwise reject.

To prove E_{LBA} is undecidable again via A_{TM}

Detour via <u>Computation Histories</u>

A CH for a TM M on a string w is a string

$\# C_1 \# C_2 \# \cdots \# C_k \#$ where

1. C_1 is the starting configuration of M on w.
2. C_k is an accepting configuration for M
3. for $1 \le i < k$ C_i yields C_{i+1}

Given M, w define

$$CH_{\langle M,w\rangle} = \{ x \mid x \text{ is a CH for } M \text{ on } w \}$$

$$CH_{\langle M,w\rangle} = \phi \quad \text{if } M \text{ does not accept } w$$

$$\neq \phi \quad \text{if } M \text{ accepts } w.$$

An LBA can recognize $CH_{\langle M,w\rangle}$.

b Given M, w there is an LBA $B_{\langle M, w\rangle}$ that can decide $CH_{\langle M, w\rangle}$

Given x

check if $x = \#C_1\#C_2 \cdots \#C_k\#$

is a valid CH for M on w.

$$\#C_1\#C_2\# \cdots \#C_k\#$$

Reduce A_{TM} to E_{LBA}

$E_{LBA} = \{ \langle M \rangle \mid M$ is a LBA
$L(M) = \phi \}$

- Say E_{LBA} has a decider H

Create decider N for A_{TM}

N: on input $\langle M, w \rangle$

1. generate $\langle B_{\langle M, w \rangle} \rangle$
2. Run H on $\langle B_{\langle M, w \rangle} \rangle$
3. If H accepts, reject
4. If H rejects, accept

H accepts $\langle B_{\langle M, w \rangle} \rangle$ iff

$CH_{\langle M, w \rangle} = \phi$ iff

M does not accept w

$\overline{CH}_{\langle M,w\rangle}$ is a CFL (not true exactly)

$= \{ x \mid x$ is not a ~~complete~~ CH for M on $w\}$

$\Rightarrow \exists$ a CFG. $G_{\langle M,w\rangle}$ that generates $\overline{CH}_{\langle M,w\rangle}$

$ALL_{CFG} = \{ \langle G \rangle \mid G$ is a CFG
$L(G) = \Sigma^* \}$

is undecidable.

~~Given~~ Assume H decides
ALL_{CFG}

give N for A_{TM}

N: on input $\langle M, w \rangle$

1. generate $\langle G_{\langle M,w \rangle} \rangle$
2. Run H on $\langle \; G_{\langle M,w \rangle} \; \rangle$
3. Accept if H rejects
4. Reject if H accepts

$\overline{CH}_{\langle M,w\rangle}$ is a CFL (sort of)

$= \{ x \mid x$ is not a CH for $\langle M, w \rangle \}$

$x \in \overline{CH}_{\langle M,w\rangle}$ if

1. x is badly formed or
2. C_1 is not a starting conf for M or
3. C_k is not an accepting
4. or $\exists\, i$ s.t. C_i does not yield C_{i+1}

$\exists$ a PDA for $\overline{CH}_{\langle M, w \rangle}$

① ② ③ easy !

④ is the hard part

$\# C_1 \# C_2 \# C_3 --- \# C_k \#$

non-deterministically guess i and check that C_i <u>does not</u> yield C_{i+1}.

$C_i = \alpha_i q_i \beta_i$

$C_{i+1} = \alpha_{i+1} q_{i+1} \beta_{i+1}$

$--- \# C_i \# C_{i+1} \# ---$

Revised defn of CH

$\# C_1 \# C_2 --- \# C_k \#$

but $\# C_1 \# C_2^R \# C_3 \# C_4^R \# --- \# C_k \#$

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push C_i onto stack

check that C_{i+1}^R is not yielded by C_i