CS 273, Fall 2006

Final Exam

December 14, 2006

INSTRUCTIONS (read carefully)

- Fill in the following information giving name and netID.

NAME:

NETID:

- CLEARLY print your name and NETID on every page.
- The exam contains 9 pages and 8 problems. Make sure you have a complete exam.
- The point value of each problem is indicated next to the problem and below.
- You have three hours.
- It is wise to skim all the problems before beginning, to best plan your time.
- This is a closed book exam. No notes of any kind are allowed. Do all work in the space provided, using the backs of sheets if necessary. See the proctor if you need more paper.
- Please bring apparent bugs to the attention of the proctors.

Problem	Possible	Score
1	18	
2	20	
3	12	
4	10	
5	10	
6	10	
7	10	
8	10	
Total	100	

Problem 1: True/False (18 points)

Completely write out "True" if the statement is necessarily true. Otherwise, completely write "False". Other answers (e.g. "T") will receive credit only if your intent is unambiguous. For example, "$x + y > x$" has answer "False" assuming that y could be 0 or negative. But "If x and y are natural numbers, then $x + y \geq x$" has answer "True". You do not need to explain or prove your answers.

1. The language $L = \{0^n 1^n \mid n \geq 0\}$ is Turing decidable.
2. The language $L = \{0^p \mid p \text{ is prime }\}$ is Turing decidable.
3. If you exchange the final and non-final state assignments for a nondeterministic finite automata, it recognizes the complement of the original language.
4. The set of all rational numbers (of the form a/b where $a, b \in \mathbb{N}_0$, $b \neq 0$) is countable.
5. The membership problem for Turing machines is Turing decidable.
6. The language $L = \{0^n \mid n = (k^{k/3}) \bmod 8, \text{ for some } k \in \mathbb{N} \}$ is regular.
7. The language $\{0^n \mid n = k! \text{ for some } k \in \mathbb{N}\}$ is regular. Remember that n! denotes the factorial of n.
8. For any DFA A, there is some $n \in \mathbb{N}$, such that A will halt within n steps on all inputs.
9. Some context-free languages are regular.
10. Context-free languages are closed under intersection.
11. The description of any Turing machine is finite.
12. Turing machines extended to have access to a stack, in addition to the usual tape, can decide more languages than normal Turing machines.
13. If a DFA M rejects all input strings, then $L(M) = \{\emptyset\}$.
14. The language of all encodings of Turing machines M that halt on the blank tape is Turing recognizable.
15. If a language L satisfies the Pumping Lemma for regular languages (as studied in this course), then L is regular.
16. The membership problem for context-free grammars is decidable.
17. The transition function for a pushdown automaton is a function from $Q \times \Sigma_\epsilon \times \Gamma_\epsilon$ to $Q \times \Gamma_\epsilon$, where Q is the set of states, Σ is the input alphabet, and Γ is the stack alphabet.
18. $E_{DFA} = \{< M >: M \text{ is a DFA and } L(M) = \emptyset\}$ is decidable.

Problem 2: Classification (20 points)

For each language L described below, classify L as

- **R**: Any language satisfying the information must be regular.
- **C**: Any language satisfying the information must be context-free, but not all languages satisfying the information are regular.
- **DEC**: Any language satisfying the information must be decidable, but not all languages satisfying the information are context-free.
- **NONDEC**: Not all languages satisfying the information are decidable. (Some might be only Turing recognizable or perhaps even not Turing recognizable.)

For each language, circle the appropriate choice (**R**, **C**, **DEC**, or **NONDEC**). If you change your answer be sure to erase well or otherwise make your final choice clear. **Ambiguously marked answers will receive no credit.**

1. **R C DEC NONDEC**
   $L = \{uv \mid u, v \in \{0,1\}^*,$ the number of 1's in u is equal to the number of 1's in v $\}$.
2. **R C DEC NONDEC**
   $L = \{0^i w 1^i | w \in \{0,1\}^*, i \geq 2\}$.
3. **R C DEC NONDEC**
   $L = \{0^i 1^j \mid 2i + 4j = 273,$ i and j are positive integers $\}$.
4. **R C DEC NONDEC**
   $L = \{0^i 1^j 2^k \mid i, j, k$ are distinct positive integers$\}$.
5. **R C DEC NONDEC**
   $L = \{\langle M \rangle \mid M$ is a Turing machine and M accepts at least 273 strings $\}$.
6. **R C DEC NONDEC**
   $L = \{\langle M \rangle \mid M$ is a Turing machine and M halts on every input of length less than 273$\}$.
7. **R C DEC NONDEC**
   $L = \{n \mid$ there are less than n students enrolled in UIUC in Fall 2006$\}$.
8. **R C DEC NONDEC**
   $L = A \cup B$, where A is regular and B is a CFL.
9. **R C DEC NONDEC**
   $L = A \cap B$, where A and B are CFLs.
10. **R C DEC NONDEC**
    L is a subset of L', where L' is a CFL.

Problem 3: Short answer (12 points)

(a) Give an example of an ambiguous grammar G. Explain why your grammar is indeed ambiguous, using a specific string w.

(b) Explain briefly how a normal Turing machine can simulate a Turing machine whose tape is infinite in both directions. (Just give the main ideas. Don't go into detail.)

(c) State Rice's Theorem.

Problem 4: Writing a proof (10 points)

Prove that a language L is decidable if and only if L is Turing recognizable and $\overline{L}$ is Turing recognizable. You can assume that we have already shown that the complement of a decidable language is decidable.

Problem 5: Pumping lemma (10 points)

Let $\Sigma = \{0, 1, 2\}$. Let $L = \{2ww2 : w \in \{0, 1\}^*\}$. Prove that L is not regular by filling in the missing parts of the following pumping lemma proof.

Suppose that L were regular. Let p be the constant given by the pumping lemma.

Consider the string $w_p =$

Because $|w_p| \geq p$, there must exist strings x, y, and z such that $w_p = xyz$, $|xy| \leq p$, $|y| > 0$, and $xy^i z \in L$ for every $i \geq 0$.

Therefore, we have a contradiction and so L must not have been regular.

Problem 6: PDA design (10 points)

Let $\Sigma = \{0, 1\}$. Let $L = \{xy\#y'x^R | x, y \in \Sigma^*, y' = (y^R)^C\}$, where x^C denotes the bitwise complement of x (for example, $0001101 = (1110010)^C$) and x^R is the reverse of the string x. So, an example of a string in L is 011110#100110 because 100 is the reversed complement of 110 and 110 is the reverse of 011.

Design a PDA whose language is L. Present your PDA as a state diagram and explain briefly how it works.

Problem 7: NFA reversal (10 points)

Assume we have the NFA $N = (Q, \Sigma, \delta, q_0, \{q_f\})$. Note that N has only a single final state, q_f. We want to design a new NFA $N' = (Q', \Sigma, \delta', q_0', F)$, such that $L(N') = (L(N))^R$. Remember that L^R is the set of the reversed versions of all strings in L. That is, $L^R = \{x^R : x \in L\}$.

(a) Explain the basic idea behind the construction.

(b) Give an example of an NFA N and the corresponding NFA N' accepting the reversed language. Your example N must have no more than four states and must not be deterministic.

(b) Having the above solution in mind, express each of Q', δ', q_0' and F in terms of Q, δ, q_0, and q_f, using sound mathematical notation.

Problem 8: Reduction (10 points)

Let $T = \{\langle M\rangle \mid \text{M is a Turing machine whose language is } \Sigma^* \text{ where } \Sigma \text{ is M's alphabet}\}$. Show that T is undecidable using a reduction from A_{TM} (Turing machine membership). You may not use Rice's Theorem.