CS 473UG: Combinatorial Algorithms, Spring 2006
Homework 7

Due Tuesday, April 11, 2006, at the beginning of class

1. A UIUC graduate student was trying to make ends meet on a very small stipend. He went to the library and looked up the National Research Council's publication entitled "Recommended Dietary Allowances" and was able to determine a minimum daily intake quantity of each essential nutrient for a male in his weight and age category. Let m denote the number of nutrients that he identified as important to his diet, and let b_i for $i = 1, 2, \ldots, m$ denote his personal minimum daily requirements. Next, he made a list of his favorite foods (which, except for pizza and due mostly to laziness and ineptitude in the kitchen, consisted almost entirely of frozen prepared meals). He then went to the local grocery store and made a list of the unit price for each of his favorite foods. Let us denote these prices as c_j for $j = 1, 2, \ldots, n$. In addition to prices, he also looked at the labels and collected information about how much of the critical nutrients are contained in one serving of each food. Let us denote by $a_{i,j}$ the amount of nutrient i contained in food j. (Fortunately, he was able to call his favorite pizza delivery service and get similar information from them.)

   (a) In terms of this information, formulate a linear programming version of this problem.

   (b) Formulate the dual to the linear programming problem from part a.

2. For each of problems **a, b, c**, reduce (in polynomial time) one of problems **d, e, f** to it. Make sure to analyze the running time of the reduction.

   (a) **Set Packing**
   Instance: Collection C of finite sets, positive integer $K \leq |C|$.
   Question: Does C contain K disjoint sets?

   (b) **Minimum Sum of Squares**
   Instance: Finite set A, "size" $s(a) \in \mathbb{Z}^+$ for each $a \in A$, positive integers K and J.
   Question: Can the elements of A be partitioned into K disjoint sets $A_1, A_2, \ldots, A_k$ such that

$$\sum_{i=1}^{k} \left(\sum_{a \in A_i} s(a) \right)^2 \leq J?$$

   (c) **Partition into Hamiltonian Subgraphs**
   Instance: Graph $G = (V, E)$, positive integer $K \leq |V|$.
   Question: Can the vertices of G be partitioned into $k \leq K$ disjoint set $V_1, V_2, \ldots, V_k$ such that, for $1 \leq i \leq k$, the subgraph induced by V_i contains a Hamiltonian circuit?

   (d) **Hamiltonian Path**
   Instance: A graph $G = (V, E)$. Question: Does G contain a Hamiltonian path, that is, an ordering $\langle v_1, v_2, \ldots, v_n \rangle$ of the vertices of G, where $n = |V|$, such that $\{v_i, v_{i+1} \in E$ for all i, $1 \leq i < n$?

   (e) **Clique**
   Instance: A graph $G = (V, E)$ and a positive integer $J \leq |V|$. Question: Does G contain a *clique* of size J or more, that is, a subset $V' \subseteq V$ such that $|V'| \geq J$ and every two vertices in $|V'|$ are joined by an edge in E?

   (f) **Partition**
   Instance: A finite set A and a "size" $s(a) \in \mathbb{Z}^+$ for each $a \in A$.
   Question: Is there a subset $A' \subseteq A$ such that

$$\sum_{a \in A'} s(a) = \sum_{a \in A - A'} s(a)?$$

3. Look at the description of the hitting set problem, described in problem 5 from chapter 8, which is on page 506 of the textbook. Give a polynomial time reduction from one of the problems introduced in section 8.1 of the textbook - namely, independent set, vertex cover, set cover, or set packing - to the hitting set problem.

   Note that you only have to do ONE reduction here, and you get to pick which problem you want to reduce from.