

CS 473UG: Combinatorial Algorithms, Spring 2006

Homework 0

Due Tuesday, January 24, at the beginning of class (11:00am CDT)

Name:	
Net ID:	Alias:

I understand the Homework Instructions and FAQ.

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- Neatly print your full name, your NetID, and an alias of your choice in the boxes above. Grades will be listed on the course web site by alias. Please write the same alias on every homework and exam! For privacy reasons, your alias should not resemble your name or NetID. By providing an alias, you agree to let us list your grades; if you do not provide an alias, your grades will not be listed. ***Never give us your Social Security number!***
 - Read the “Homework Instructions and FAQ” on the course web page, and then check the box above. This page describes what we expect in your homework solutions—start each numbered problem on a new sheet of paper, write your name and NetID on every page, don’t turn in source code, analyze and prove everything, use good English and good logic, and so on—as well as policies on grading standards, regrading, and plagiarism. **See especially the course policies regarding the magic phrases “I don’t know” and “and so on”.** If you have *any* questions, post them to the course newsgroup or ask during lecture.
 - Don’t forget to submit this cover sheet with the rest of your homework solutions.
 - This homework tests your familiarity with prerequisite material—big-Oh notation, elementary algorithms and data structures, recurrences, discrete probability, and most importantly, induction—to help you identify gaps in your knowledge. **You are responsible for filling those gaps on your own.** Chapters 1–10 of CLRS should be sufficient review, but you may also want consult your discrete mathematics and data structures textbooks.
1. An *Eulerian circuit* is a walk on the edges of a graph which uses each edge in the graph exactly once and ends at the same vertex where it begins. Prove that a graph has an Eulerian circuit if and only if the graph is connected and each vertex has even degree.
 2. The *fractional part* of x is the amount by which it exceeds its floor, $\lfloor x \rfloor$. For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, let $S = \{x, 2x, \dots, (n-1)x\}$.
 - (a) Prove that if some pair of numbers in S have fractional parts that differ by at most $1/n$, then some number in S is within $1/n$ of an integer.
 - (b) Use part (a) to prove that some number in S is within $1/n$ of an integer.

3. Sort the following 20 functions from asymptotically smallest to asymptotically largest, indicating ties if there are any. You do not need to turn in proofs (in fact please, *don't* turn in proofs), but you should do them anyway just for practice.

We use $\lg n$ to denote $\log_2 n$, $\lg^{1000} n$ to denote $(\lg n)^{1000}$ and $\lg^{(1000)} n$ to denote $\lg \lg \lg \dots \lg n$ where we take the log 1000 times. Finally, $\lg^* n$ denotes the number of times that we must iterate taking the \lg before the result is less than or equal to 1. For example, if $4 < n \leq 8$, then $\lg^* n = 3$. If $32 < n \leq 64$, then $\lg^* n = 6$.

$$\begin{array}{cccccc}
 1 & n & n^2 & \lg n & \lg^* n & \\
 2^{2^{\lg \lg n + 2}} & \lg^* 2^n & 2^{\lg^* n} & \lfloor \lg(n!) \rfloor & \lfloor \lg n \rfloor! & \\
 n^{\lg n} & (\lg n)^n & (\lg n)^{\lg n} & n^{1/\lg n} & n^{\lg \lg n} & \\
 \log_{1000} n & \lg^{1000} n & \lg^{(1000)} n & (1 + \frac{1}{1000})^n & n^{1/1000} &
 \end{array}$$

To simplify notation, write $f(n) \ll g(n)$ to mean $f(n) = o(g(n))$ and $f(n) \equiv g(n)$ to mean $f(n) = \theta(g(n))$. For example, the functions $n^2, n, \binom{n}{2}, n^3$ could be sorted either as $n \ll n^2 \equiv \binom{n}{2} \ll n^3$ or as $n \ll \binom{n}{2} \equiv n^2 \ll n^3$.

4. Solve the following recurrences. State tight asymptotic bounds for each function in the form $\theta(f(n))$ for some recognizable function $f(n)$. You do not need to turn in proofs, but you should do them for practice. If no base cases are given, assume something reasonable but nontrivial. Extra credit will be given for more exact solutions.

(a) $A(n) = 3A(n/2) + n$

(b) $B(n) = \max_{n/3 < k < 2n/3} (B(k) + B(n-k) + n)$

(c) $C(n) = 3\sqrt{n}C(\sqrt{n}) + n$

(d) $D(n) = 2D(n/2) + n/\lg n$

5. This problem asks you to analyze what happens you throw balls randomly into bins. There are n bins. Each time you throw a ball, it lands in a random bin. Each bin is equally likely, and the results of different throws are independent.

(a) Suppose k of the n bins are initially empty and you stop just after a ball lands in an empty bin. What is the expected number of throws? Prove your answer is correct.

(b) Now suppose initially every bin is empty, and you stop as soon as *every* bin is occupied. What is the expected number of throws? Prove your answer is correct.