CS 418: Homework #3 Solution

1. (20 points) You are given three RGB colors with corresponding alpha values: $A = (0.7, 0.5, 0.6)$, $B = (1, 1, 0)$, and $C = (0, 0, 1)$ with alpha values $\alpha_A = 0.5, \alpha_B = 0.9$, and $\alpha_C = 0.7$.

   Solution:

   For both parts below it will be useful to derive a general formula for calculating the composition of 3 colors, which we denote D, E and F.

   To compute $(D$ over E over $F)$, or $DoEoF$, we need the following two components:

$$
\begin{aligned}
(DoEoF)' &= \alpha_D D + (1 - \alpha_D)(EoF)' \\
&= \alpha_D D + (1 - \alpha_D)(\alpha_E E + (1 - \alpha_E)\alpha_F F) \\
&= \alpha_D D + (1 - \alpha_D)\alpha_E E + (1 - \alpha_D)(1 - \alpha_E)\alpha_F F \\
\alpha_{DoEoF} &= \alpha_D + (1 - \alpha_D)(\alpha_{EoF}) \\
&= \alpha_D + (1 - \alpha_D)(\alpha_E + (1 - \alpha_E)\alpha_F) \\
&= \alpha_D + (1 - \alpha_D)\alpha_E + (1 - \alpha_D)(1 - \alpha_E)\alpha_F
\end{aligned}
$$

   Then we de-multiply $(DoEoF)'$ by α_{DoEoF} to get $(DoEoF)$.

   (a) Compute the color and alpha value for (A over B over C).

   Solution:

$$(AoBoC)' = (0.8, 0.7, 0.335)$$
$$\alpha_{AoBoC} = .985$$

   De-multiply, we get:

$$AoBoC = (0.812, 0.711, 0.340)$$

   (b) Compute the color and alpha value for (C over B over A).

   Solution:

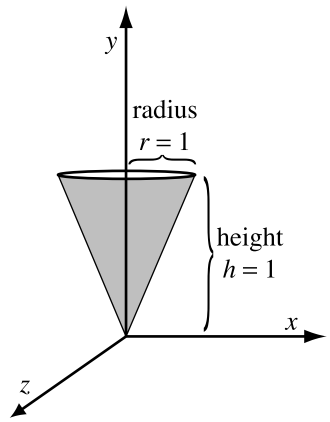
$$(CoBoA)' = (0.281, 0.278, 0.709)$$
$$\alpha_{CoBoA} = .985$$

   De-multiply, we get:

$$CoBoA = (0.285, 0.282, 0.720)$$

   We can see that in general, $AoBoC$ is not the same as $CoBoA$.

2. (20 points) Consider the (uncapped) unit cone whose apex is the origin and whose radius and height are both 1.



(a) What is the parametric form of this surface? Make sure to indicate what your parameter values mean and what their ranges are.

Solution:

$$\begin{cases} x = r\cos\theta \\ y = r \\ z = r\sin\theta \end{cases}$$

with $0 \leq \theta < 2\pi, 0 \leq r \leq 1$.

(b) Briefly describe how to generate this object as a generalized cylinder.

Solution:

To define a generalized cylinder, we need to specify its cross section and scaling function. For this cone example, using $0 \leq r \leq 1$ as the parameter, the cross section is circle $x^2 + z^2 = r^2$, and scaling function is $s(r) = r$.

(c) What is the implicit form of this cone? You can assume that the coordinates will always be restricted to the range $0 \leq x, y, z \leq 1$.

Solution:

$x^2 + z^2 - y^2 = 0$

3. (20 points) Suppose we want to construct a model of an ellipsoid centered at the origin, whose axes are the x, y, z coordinate axes and whose radii are a, b, c, respectively.

(a) Write an implicit equation for this ellipsoid.

Solution:

$$F(x, y, z) = (\frac{x}{a})^2 + (\frac{y}{a})^2 + (\frac{z}{a})^2 - 1 = 0$$

(b) Convert this equation into the standard quadratic form $\mathbf{v}^\top \mathbf{Q} \mathbf{v} = 0$.

Solution:

We have squared terms for x, y, z, and a constant. When put into a matrix $\mathbf{Q}$, they fir in the diagonals so we get the correct terms when multiplied with $\mathbf{v}^\top$ and $\mathbf{v}$.

$$F(x,y,z) = \mathbf{v}^\mathsf{T}\mathbf{Q}\mathbf{v} = [\,x \quad y \quad z \quad 1\,] \begin{bmatrix} 1/a^2 & & & \\ & 1/b^2 & & \\ & & 1/c^2 & \\ & & & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0$$

(c) Show how to derive this matrix $\mathbf{Q}$ via a transformation of the quadratic form for the unit sphere.

Solution:

The implicit equation for the unit sphere is: $F(x,y,z) = x^2 + y^2 + z^2 - 1 = 0$

Again, in order for the sphere's quadratic form to multiply out to the same results as the implicit equation, we need to put the coefficients of its terms in the diagonal where each respective dimension is squared.

$$F(x,y,z) = \mathbf{v}^\mathsf{T}\mathbf{Q}\mathbf{v} = [\,x \quad y \quad z \quad 1\,] \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = 0$$

To scale it so that the x, y, z have radii a, b, c, we apply the following scale matrix that scales it by a, b, c, since the original radius is 1.

$$S = \begin{bmatrix} a & & & \\ & b & & \\ & & c & \\ & & & 1 \end{bmatrix}$$

To scale it to an ellipsoid, we use the equation:

$$\mathbf{Q}_{ellipsoid} = (S^{-1})^\mathsf{T}\mathbf{Q}_{sphere}S^{-1} = \begin{bmatrix} 1/a & & & \\ & 1/b & & \\ & & 1/c & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} \begin{bmatrix} 1/a & & & \\ & 1/b & & \\ & & 1/c & \\ & & & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1/a^2 & & & \\ & 1/b^2 & & \\ & & 1/c^2 & \\ & & & -1 \end{bmatrix}$$

4. (20pts) Suppose that you are given a point $\mathbf{p} = (x,y,z)$ and a plane $\mathbf{n}\cdot\mathbf{x} + d = 0$ where $\|\mathbf{n}\| = 1$.

(a) Let $\mathbf{q}$ be the perpendicular projection of $\mathbf{p}$ onto the plane. Derive an expression of the location $\mathbf{q}$ in terms of $\mathbf{p}$, $\mathbf{n}$, and d.

Solution:

Let us draw a ray from $\mathbf{p}$ perpendicular to the plane. Then point $\mathbf{q} = \mathbf{p} - t\mathbf{n}$ for some t (defining $\mathbf{q} = \mathbf{p} + t\mathbf{n}$ will also work, since the sign difference will cancel out anyway).

Like ray-plane intersection, we substitute ray equation into plane to solve for t:
$\mathbf{n}\cdot(\mathbf{p} - t\mathbf{n}) + d = 0$
$\mathbf{n}\cdot\mathbf{p} - t(\mathbf{n}\cdot\mathbf{n}) + d = 0$

$t = (\mathbf{n} \cdot \mathbf{p} + d)/(\mathbf{n} \cdot \mathbf{n}) = (\mathbf{n} \cdot \mathbf{p} + d)$

After finding t, we plug it back into the ray equation:
$\mathbf{q} = \mathbf{p} - (\mathbf{n} \cdot \mathbf{p} + d)\mathbf{n}$.

(b) Derive the equation for the location of the point $\mathbf{p}$ projected onto the plane through the origin $(0, 0, 0)$ as pictured in the figure below.

Solution:

Let $\mathbf{q}$ be this location. Then $\mathbf{q} = t\frac{\mathbf{p}}{\|\mathbf{p}\|}$ since we are projecting from the origin through $\mathbf{p}$. Again, plug into the plane equation to solve for t:
$\mathbf{n} \cdot (t\frac{\mathbf{p}}{\|\mathbf{p}\|}) + d = 0$
$t(\frac{\mathbf{p} \cdot \mathbf{n}}{\|\mathbf{p}\|}) = -d$
$t = (\frac{-d\|\mathbf{p}\|}{\mathbf{p} \cdot \mathbf{n}})$

Put t back into the ray equation:
$\mathbf{q} = (\frac{-d\|\mathbf{p}\|\mathbf{p}}{(\mathbf{p} \cdot \mathbf{n})\|\mathbf{p}\|}) = \frac{-d\mathbf{p}}{\mathbf{p} \cdot \mathbf{n}}$

(c) Suppose that we now want to write some code to do projective shadows as discussed in class. Using the equation you derived in the previous part, detail how you would support projection onto a plane through any arbitrary point (not just the origin).

Solution:

Given an arbitrary point $\mathbf{r}$, we would first translate the plane and point $\mathbf{p}$ by $-\mathbf{r}$ so that we are projecting from the origin. Use the above equation to find the projection point, then translate the resulting point by $\mathbf{r}$ to obtain the final answer.

Alternatively, we can obtain an exact equation if we take into account projecting from an arbitrary point $\mathbf{r}$. Then $\mathbf{q} = \mathbf{r} + t\frac{\mathbf{p}-\mathbf{r}}{\|\mathbf{p}-\mathbf{r}\|}$ for some t, since we are casting a ray from the point $\mathbf{r}$ through point $\mathbf{p}$.

Plug into plane equation to solve for t:
$\mathbf{n} \cdot (\mathbf{r} + t\frac{\mathbf{p}-\mathbf{r}}{\|\mathbf{p}-\mathbf{r}\|}) + d = 0$
$t(\frac{(\mathbf{p}-\mathbf{r}) \cdot \mathbf{n}}{\|\mathbf{p}-\mathbf{r}\|}) = -d - \mathbf{n} \cdot \mathbf{r}$
$t = \frac{(-d-\mathbf{n} \cdot \mathbf{r})\|\mathbf{p}-\mathbf{r}\|}{(\mathbf{p}-\mathbf{r}) \cdot \mathbf{n}}$

Put t back into the ray equation:
$\mathbf{q} = \mathbf{r} + (\frac{(-d-\mathbf{n} \cdot \mathbf{r})\|\mathbf{p}-\mathbf{r}\|}{(\mathbf{p}-\mathbf{r}) \cdot \mathbf{n}})\frac{\mathbf{p}-\mathbf{r}}{\|\mathbf{p}-\mathbf{r}\|} = \mathbf{r} + \frac{(-d-\mathbf{n} \cdot \mathbf{r})}{\mathbf{n}}$

Full credit is given for both approaches.

5. (20pts) A quadric surface is described by the implicit equation $\mathbf{x}^{\mathsf{T}}\mathbf{Q}\mathbf{x} = 0$ where $\mathbf{x}$ is the homogeneous column vector $(x_1, x_2, x_3, 1)$ and $\mathbf{Q}$ is a symmetric 4×4 matrix. Suppose you are given a ray $\mathbf{p} + t\mathbf{d}$ with $\|\mathbf{d}\| = 1$.

(a) Derive the equation for the value(s) of t at which the ray will intersect this surface.

Solution:

We can substitute $\mathbf{x}$ in the implicit equation with the equation of the ray: $(\mathbf{p} + t\mathbf{d})^T\mathbf{Q}(\mathbf{p} + t\mathbf{d}) = 0$
Using the rule $\mathbf{x}^T\mathbf{Q}\mathbf{x} = \mathbf{x} \cdot \mathbf{Q}\mathbf{x}$, we can eliminate the transpose: $(\mathbf{p} + t\mathbf{d}) \cdot \mathbf{Q}(\mathbf{p} + t\mathbf{d}) = 0$
Multiply into terms of t: $\mathbf{p} \cdot \mathbf{Q}(\mathbf{p} + t\mathbf{d}) + (t\mathbf{d}) \cdot \mathbf{Q}(\mathbf{p} + t\mathbf{d}) = 0$
$\mathbf{p} \cdot \mathbf{Q}\mathbf{p} + \mathbf{p} \cdot t\mathbf{Q}\mathbf{d} + (t\mathbf{d}) \cdot \mathbf{Q}\mathbf{p} + (t\mathbf{d}) \cdot t\mathbf{Q}\mathbf{d} = 0$

$(\mathbf{d} \cdot \mathbf{Qd})t^2 + (\mathbf{p} \cdot \mathbf{Qd} + \mathbf{d} \cdot t\mathbf{Qp})t + p \cdot \mathbf{Qp} = 0$

Since $\mathbf{Q}$ is symmetric, $\mathbf{x} \cdot \mathbf{Qy} = \mathbf{y} \cdot \mathbf{Qx}$ (Or if you chose not to eliminate transposes, $\mathbf{x}^T\mathbf{Qy} = \mathbf{y}^T\mathbf{Qx}$). However, note that $\mathbf{x} \cdot \mathbf{Qy} \neq (\mathbf{x} \cdot \mathbf{y})\mathbf{x}$. We then simplify to: $(\mathbf{d} \cdot \mathbf{Qd})t^2 + 2(\mathbf{p} \cdot \mathbf{Qd})t + p \cdot \mathbf{Qp} = 0$

This is in a quadratic form, where $a = \mathbf{d} \cdot \mathbf{Qd}$, $b = 2(\mathbf{p} \cdot \mathbf{Qd})$, $c = p \cdot \mathbf{Qp}$, and $t = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.

(b) A given ray may intersect a quadric at 0, 1, or 2 spots. Derive an expression which can be used to test whether the ray intersects the surface or whether it misses it entirely.

Solution:

If the discriminant, $b^2 - 4ac$, equals 0, and the resulting $t \geq 0$, the ray hits in one spot. If the discriminant is greater than 0, there is a hit for every positive t (up to two hits). Otherwise the ray misses.