

CS 418: Homework #1

Solution

1. (15pt)

(a) Let $\mathbf{v} = [5 \ 3 \ 7]$. Compute $\|\mathbf{v}\|$, the vector length of \mathbf{v} .

Solution:

$$\|\mathbf{v}\| = \sqrt{5^2 + 3^2 + 7^2} = \sqrt{83} \approx 9.11$$

(b) Let $\mathbf{u} = [2 \ 3 \ 1]$ and $\mathbf{v} = [5 \ 3 \ 7]$. Compute the angle \mathbf{u} makes with \mathbf{v} . Express your answer in radians.

Solution:

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| * \|\mathbf{v}\|} = \frac{26}{\sqrt{1162}}$$
$$\theta = \cos^{-1} \left(\frac{26}{\sqrt{1162}} \right) = \cos^{-1}(0.7627) = 0.703 \text{ radian}$$

(c) Let $\mathbf{u} = [2 \ 3 \ 1]$ and $\mathbf{v} = [5 \ 3 \ 7]$. Find a unit vector perpendicular to both \mathbf{u} and \mathbf{v} .

Solution:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ 5 & 3 & 7 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 3 & 7 \end{vmatrix} i - \begin{vmatrix} 2 & 1 \\ 5 & 7 \end{vmatrix} j + \begin{vmatrix} 2 & 3 \\ 5 & 3 \end{vmatrix} k = [18 \ -9 \ -9]$$

Unitize $\mathbf{u} \times \mathbf{v}$, the result is $\left[\frac{2}{\sqrt{6}} \ -\frac{1}{\sqrt{6}} \ -\frac{1}{\sqrt{6}} \right]$

(d) Given a vector $\mathbf{v} = [x \ y]$, show that the vector $\mathbf{u} = [-y \ x]$ is orthogonal (i.e., perpendicular) to \mathbf{v} .

Solution:

Two vectors \mathbf{u}, \mathbf{v} are perpendicular iff $\mathbf{u} \cdot \mathbf{v} = 0$, and $[x \ y] \cdot [-y \ x] = x(-y) + yx = 0$

2. (15pt) Suppose you are given matrices

$$\mathbf{M} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 4 \\ 0 & -4 & 3 \end{bmatrix},$$

$$\mathbf{N} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

and vectors $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

(a) Compute the determinant $\det \mathbf{M}$.

Solution:

$$\det \mathbf{M} = 1 \cdot \begin{vmatrix} 3 & 4 \\ -4 & 3 \end{vmatrix} - 0 \cdot \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} + 0 \cdot \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ -4 & 3 \end{vmatrix} = 25$$

(b) Compute the inverse \mathbf{M}^{-1} .

Solution:

We use elementary row operations to find the inverse of \mathbf{M} .

$$\begin{aligned} & \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 3 & 4 & 0 & 1 & 0 \\ 0 & -4 & 3 & 0 & 0 & 1 \end{array} \right] \\ \Rightarrow & \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4/3 & 0 & 1/3 & 0 \\ 0 & -4 & 3 & 0 & 0 & 1 \end{array} \right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -5/3 & 1 & -2/3 & 0 \\ 0 & 1 & 4/3 & 0 & 1/3 & 0 \\ 0 & -4 & 3 & 0 & 0 & 1 \end{array} \right] \\ \Rightarrow & \left[\begin{array}{ccc|ccc} 1 & 0 & -5/3 & 1 & -2/3 & 0 \\ 0 & 1 & 4/3 & 0 & 1/3 & 0 \\ 0 & 0 & 25/3 & 0 & 4/3 & 1 \end{array} \right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -5/3 & 1 & -2/3 & 0 \\ 0 & 1 & 4/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & 0 & 4/25 & 3/25 \end{array} \right] \\ \Rightarrow & \left[\begin{array}{ccc|ccc} 1 & 0 & -5/3 & 1 & -2/3 & 0 \\ 0 & 1 & 0 & 0 & 3/25 & -4/25 \\ 0 & 0 & 1 & 0 & 4/25 & 3/25 \end{array} \right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2/5 & 1/5 \\ 0 & 1 & 0 & 0 & 3/25 & -4/25 \\ 0 & 0 & 1 & 0 & 4/25 & 3/25 \end{array} \right] \\ \text{So } \mathbf{M}^{-1} &= \begin{bmatrix} 1 & -2/5 & 1/5 \\ 0 & 3/25 & -4/25 \\ 0 & 4/25 & 3/25 \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 25 & -10 & 5 \\ 0 & 3 & -4 \\ 0 & 4 & 3 \end{bmatrix} \end{aligned}$$

(c) Compute matrix-matrix multiplication \mathbf{MN} .

Solution:

$$\mathbf{MN} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 4 \\ 0 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 1 \\ 7 & 7 & 4 \\ -1 & -1 & 3 \end{bmatrix}$$

(d) Compute matrix-vector multiplication \mathbf{Mv} .

Solution:

$$\mathbf{Mv} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 4 \\ 0 & -4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y + z \\ 3y + 4z \\ -4y + 3z \end{bmatrix}$$

(e) Compute the multiplication \mathbf{uv}^T .

Solution:

$$\mathbf{uv}^T = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} [x \ y \ z] = \begin{bmatrix} 2x & 2y & 2z \\ 3x & 3y & 3z \\ x & y & z \end{bmatrix}$$

3. (15pt) Suppose you are given the following four polynomials:

$$b_1(x) = (1 - x)^3$$

$$b_2(x) = 3x(1 - x)^2$$

$$b_3(x) = 3x^2(1 - x)$$

$$b_4(x) = x^3,$$

and the following two vectors $\mathbf{B} = \begin{bmatrix} b_1(x) \\ b_2(x) \\ b_3(x) \\ b_4(x) \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix}$. There exists a matrix \mathbf{M} that satisfies the equation,

$\mathbf{B} = \mathbf{M}\mathbf{v}$. Compute \mathbf{M} .

Solution:

$$b_1(x) = (1 - x)^3 = 1 - 3x + 3x^2 - x^3 = [1 \ -3 \ 3 \ -1] \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix}$$

$$b_2(x) = 3x(1 - x)^2 = 3x - 6x^2 + 3x^3 = [0 \ 3 \ -6 \ 3] \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix}$$

$$b_3(x) = 3x^2(1 - x) = 3x^2 - 3x^3 = [0 \ 0 \ 3 \ -3] \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix}$$

$$b_4(x) = x^3 = [0 \ 0 \ 0 \ 1] \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix}$$

So

$$\mathbf{B} = \begin{bmatrix} b_1(x) \\ b_2(x) \\ b_3(x) \\ b_4(x) \end{bmatrix} = \begin{bmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4. (15pt) Suppose you are given the following matrix,

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix},$$

and vectors $\mathbf{v} = \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix}$, $\mathbf{G} = \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \mathbf{g}_3 \\ \mathbf{g}_4 \end{bmatrix}$, where $\mathbf{g}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{g}_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{g}_3 = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}$, $\mathbf{g}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix}$. Compute $\mathbf{v}^T \mathbf{M} \mathbf{G}$.

Solution:

$$\mathbf{v}^T \mathbf{M} = [1 \quad x \quad x^2 \quad x^3] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix} = [1 - 3x^2 + 2x^3 \quad 3x^2 - 2x^3 \quad x - 2x^2 + x^3 \quad -x^2 + x^3]$$

So

$$\mathbf{v}^T \mathbf{M} \mathbf{G} = [1 - 3x^2 + 2x^3 \quad 6x^2 - 4x^3 \quad 3x - 6x^2 + 3x^3 \quad -4x^2 + 4x^3]$$

5. (10pt) Suppose you are given the 2-D points $\mathbf{p}_1 = [x_0 \ y_0]$ and $\mathbf{p}_2 = [x_1 \ y_1]$. Show that the equation of the line between them is $(y_1 - y_0)x - (x_1 - x_0)y = y_1x_0 - x_1y_0$.

Solution:

Suppose the line equation is $ax + by + c = 0$, and we know that $\begin{bmatrix} a \\ b \end{bmatrix}$ is perpendicular to the line. Since vector $\mathbf{p}_1 - \mathbf{p}_2 = \begin{bmatrix} x_0 - x_1 \\ y_0 - y_1 \end{bmatrix}$ is along the line, we know from question 1(d) that $\begin{bmatrix} y_1 - y_0 \\ x_0 - x_1 \end{bmatrix}$ is a vector perpendicular to the line. We can use it as $\begin{bmatrix} a \\ b \end{bmatrix}$, which means $a = y_1 - y_0$ and $b = x_0 - x_1$.

Next we plug in \mathbf{p}_0 to the line equation to get c . From $ax_0 + by_0 + c = 0$ we have $(y_1 - y_0)x_0 + (x_0 - x_1)y_0 + c = 0$, so $c = x_1y_0 - x_0y_1$. This gives us the equation in the problem statement.

6. (15pt) The equation of a plane in 3-D space is $ax + by + cz + d = 0$. Given three points $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ find the coefficients a, b, c for the plane containing the three given points.

Solution:

We follow a similar approach to the previous question. First, we compute a normal vector for the plane. Recall that given two vectors in the plane, their cross product will be perpendicular to the plane. Thus a normal vector is:

$$\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = (\mathbf{p}_2 - \mathbf{p}_1) \times (\mathbf{p}_3 - \mathbf{p}_1)$$

Given this normal and a point in the plane, we can solve for d :

$$\begin{aligned} \mathbf{n} \cdot \mathbf{p}_1 + d &= 0 \\ d &= -\mathbf{n} \cdot \mathbf{p}_1 \end{aligned}$$

7. (15pt) Let the matrix $\mathbf{M}(\theta)$ be defined as follows:

$$\mathbf{M}(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

- (a) A square matrix \mathbf{M} is said to be *orthogonal* if and only if its inverse exists and $\mathbf{M}^{-1} = \mathbf{M}^T$. Show that $\mathbf{M}(\theta)$ is orthogonal.

Solution:

To show that $\mathbf{M}^{-1} = \mathbf{M}^T$, we simply need to show that $\mathbf{M}\mathbf{M}^T = \mathbf{I}$:

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} \cos^2 \theta + \sin^2 \theta & \cos \theta \sin \theta - \sin \theta \cos \theta \\ \sin \theta \cos \theta - \cos \theta \sin \theta & \sin^2 \theta + \cos^2 \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- (b) Show that $\mathbf{M}(\theta_1)\mathbf{M}(\theta_2) = \mathbf{M}(\theta_1 + \theta_2)$.

Solution:

$$\begin{aligned} \mathbf{M}(\theta_1)\mathbf{M}(\theta_2) &= \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{bmatrix} \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 \\ \sin \theta_2 & \cos \theta_2 \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 & -(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2) \\ \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2 & \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{bmatrix} = \mathbf{M}(\theta_1 + \theta_2) \end{aligned}$$

- (c) Show that, for all possible values of θ , the inverse of $\mathbf{M}(\theta)$ is $\mathbf{M}(-\theta)$.

Solution:

Using the result from part (b), we know that

$$\mathbf{M}(\theta)\mathbf{M}(-\theta) = \mathbf{M}(\theta + (-\theta)) = \mathbf{M}(0) = \begin{bmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So we proved that $\mathbf{M}(\theta)$ and $\mathbf{M}(-\theta)$ are inverses of each other.