CS 475: Formal Models of Computation, Fall 2005
Midterm 2 — November 1, 2005

Instructions:

- This is a closed book exam. No notes, books, calculators, etc. are allowed.
- There are 5 questions, each worth 10 points. However, all questions may not be of equal difficulty.
- Do all the work in the space provided or on the back sheets if necessary. If some your answers cannot fit into the space provided after the question, you may use the back sheets. But please tell us where to look.
- You may state and use without proof any results proved in class or in the homeworks.
- Write your name in the space provided on the top of every page.

Name	

Problem	Maximum Points	Points Earned	Grader
1	10		
2	10		
3	10		
4	10		
5	10		
Total	50		

1. This problem is True/False. Circle T if the statement is necessarily true. Circle F if the statement is not necessarily true. Provide a one line justification for your answer. Each correct answer with the right justification earns 1 point, while a correct answer without the right justification earns only $\frac{1}{2}$ point.

   (a) If $L \subseteq 0^*$ then L is context-free.
   T **F**
   False. The language $L = \{0^p \mid p \text{ is prime}\}$ is not context-free.

   (b) Any context-free language can be accepted by a pushdown automata with only one state that accepts by empty stack.
   T **F**
   True. The translation from CFGs to PDAs results in such a one state machine.

   (c) If L is a context-free language, then so is reverse$(L) = \{w^R \mid w \in L\}$.
   T **F**
   True. Reverse the right-hand side of the productions.

   (d) $\{a^i b^j c^k d^l a^l b^k c^j d^i \mid i, j, k, l \geq 0\}$ is not context-free.
   T **F**
   False. Consider the grammar with the following rules: $S \to aSd \mid A$, $A \to bAc \mid B$, $B \to cCd \mid D$, and $D \to dDa \mid \epsilon$. This grammar generates the language.

   (e) If L_1 is a CFL and L_2 is not a CFL then $L_1 \cap L_2$ is not a CFL.
   T **F**
   False. Consider a language such that $L_2 = \overline{L_1}$, L_1 is CFL and L_2 is not CFL. Then $L_1 \cap L_2 = \emptyset$ is CFL! One example of such languages would be $L_1 = \{x|x \text{ not of the form } ww\}$, and $L_2 = \overline{L_1}$.

(f) If G is a grammar in Chomsky Normal Form, and w_1 and w_2 are strings in $L(G)$ such that $|w_1| = |w_2|$, then there exists a derivation of w_1 and a derivation w_2 that contain the same number of steps.

T **F**

True. The number of steps in a derivation of a string w_1 (or w_2) is the number of internal nodes in a derivation tree for w_1. Since the grammar is in Chomsky Normal form, the derivation trees of w_1 and w_2 are binary trees, whose leaves are the symbols in w_1 and w_2. Two binary trees with equal number of leaves have equal number of internal nodes.

(g) If L is a CFL, $\overline{L}$ is not a CFL, and R is regular, then $\overline{L \cup R}$ is not a CFL.

T **F**

False. Take $R = \Sigma^*$. Then $\overline{L \cup R} = \emptyset$ which is a CFL.

(h) If M is a deterministic pushdown automaton, then $\overline{L(M)}$ is a CFL.

T **F**

True. Consider the machine where the accept and nonaccept states of M are flipped. This new DPDA recognizes $\overline{L(M)}$. Note, this does not work if M is not deterministic!

(i) If L is a CFL and R is a regular language then $R \setminus L$ is necessarily a CFL.

T **F**

False. If we take $R = T^*$ then $R \setminus L = \overline{L}$. So if the statement were true then CFLs would be closed under complementation which we know is not true.

(j) $L = \{0^n 1^m 0^n \mid n + m \geq 2\}$ is a CFL.

T **F**

True. L is the union of 111^*, 011^*0 and $\{0^n 1^* 0^n \mid n \geq 2\}$. The first two languages are regular and the third is context free. Thus, since CFLs are closed under union, L is a CFL.

2. Let $L = \{a^n b^{2n} c^n \mid n \geq 1\}$. Either prove that L is a CFL or else prove that L is not a CFL by using the pumping lemma.

   L is not context free. The proof is by showing that L does not satisfy the pumping lemma. The argument works exactly in the same way as for the language $\{a^n b^n c^n \mid n \geq 0\}$

3. Let $L = \{a^n b^{n+m} c^m \mid n, m \geq 0\}$. Either prove that L is a CFL or else prove that L is not a CFL by using the pumping lemma.

   L is context free. The easiest way to see this is to observe that the following grammar rules generate L.

$$
\begin{aligned}
S &\rightarrow AB \\
A &\rightarrow aAb \mid \epsilon \\
B &\rightarrow bBc \mid \epsilon
\end{aligned}
$$

4. For any language L define reflect$(L) = \{ww^R \,|\, w \in L\}$, where w^R denotes the reverse of string w. Prove that if L is regular, then reflect(L) is necessarily context-free.

   Let $M = (Q, \Sigma, \delta, q_0, F)$ be the DFA recognizing L. We will construct a (nondeterministic) PDA P that recognizes reflect(L). The idea is that P will process the string in two phases. In the first phase it will simulate M on the input, and at the same time push every symbol it reads onto its stack. If it reaches an accepting state of M then it will nondeterministically guess that it has read half of the string. At this point P will move the the second phase of the processing, where all P does is read a symbol, pop the top of the stack, and check that these two are equal. If it reaches a situation where the only symbol on the stack is the bottom of the stack symbol, then it will pop it and accept by empty stack.

   The formal construction is as follows. $P = (Q', \Sigma, \Gamma, \delta', q_0', Z_0, F')$ where

   - $Q' = Q \cup \{q_{\text{pop}}\}$
   - $\Gamma = \Sigma \cup \{Z_0\}$
   - $q_0' = q_0$
   - $F' = \{q_{\text{pop}}\}$
   - And δ' is defined as follows (where $a \in \Sigma$, and $q \in Q$)

$$
\begin{array}{ll}
\delta'(q, a, b) = \{(q', ab)\} & \text{provided } \delta(q, a) = q' \\
\delta'(q, \epsilon, b) = \{(q_{\text{pop}}, b)\} & \text{provided } q \in F \\
\delta'(q_{\text{pop}}, a, b) = \{(q_{\text{pop}}, \epsilon)\} & \text{provided } a = b \\
\delta'(q_{\text{pop}}, \epsilon, Z_0) = \{(q_{\text{pop}}, \epsilon)\} &
\end{array}
$$

   It is easy to see that $N(P) = $ reflect(L).

5. Recall that a deterministic context free language (DCFL) is a language L that can be recognized by a deterministic pushdown automata by final state. Show that

$$L = \{a^m b^n c^k \mid \text{either } m \neq n \text{ or } n \neq k\}$$

is not a DCFL.

Suppose L were a DCFL. Then by observation 1(g) in this exam, we know $\overline{L}$ is also context free. Consider $L' = \overline{L} \cap a^*b^*c^*$. By the closure of CFLs when intersected with regular languages, we know that L' is also a CFL. But $L' = \{a^n b^n c^n \mid n \geq 0\}$ which was shown in class to be not context free. Thus, L cannot be a DCFL.