

**CS 573: Algorithms, Fall 2009**  
**Midterm — October 7, 2009, 7–9 PM**  
Transportation building, 103

Name:		
Net ID:	Alias:	

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- This is a closed-book, closed-notes, open-brain exam. If you brought anything with you besides writing instruments and your **handwritten**  $8\frac{1}{2}'' \times 11''$  cheat sheet, please leave it at the front of the classroom.
  - Print your name, netid, and alias in the boxes above. Print your name at the top of every page (in case the staple falls out!).
  - **You should answer *all* the questions on the exam.**
  - The last few pages of this booklet are blank. Use that for a scratch paper. Please let us know if you need more paper.
  - If your cheat sheet is not hand written by yourself, or it is photocopied, please do not use it and leave it in front of the classroom.
  - Submit your cheat sheet together with your exam. An exam without your cheat sheet attached to it will not be checked.
  - If you are NOT using a cheat sheet you should indicate it in large friendly letters on this page.
  - There are 4 questions on the exam. Each question is worth 25 points.
  - Answers containing *only* the expression: “I dont know”, will get 20% of the points of the question. If you write anything else, it would be ignored. Overall, points given for “I dont know” will not exceed 10 points.
  - Write your exam using a pen not a pencil.
  - Time limit: 120 minutes.
  - Relax. Breathe. This is just an easy, silly and stupid midterm.
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#	Score	IDK	Grader
1.			
2.			
3.			
4.			
Total			

1. NP COMPLETENESS.

[25 Points]

- (A) [5 Points] Consider the complete graph  $K_n$  over  $n$  vertices (for  $n$  being even) and a maximum cut  $(S, V(K_n) \setminus S)$  in  $K_n$ . A maximum cut is a cut that has a maximum number of edges in it.

What is the size of  $S$ ?

What is the size of  $(S, V(K_n) \setminus S)$ ?

Prove your answer. (Hint: You can use the fact that the function  $f(x) = x(\alpha - x)$  gets its maximum at  $x = \alpha/2$ .)

- (B) Prove that the following problem is NP-COMplete.

**Problem:** Weighted Max Cut

*Instance:* A graph  $G = (V, E)$ , a weight function  $w$  on the edges of  $G$  assigning each edge a non-negative weight, and a number  $U$ .

*Question:* Is there a cut  $(S, V \setminus S)$  in  $G$  with total weight  $\geq U$ ?

The weight of the cut  $(S, V \setminus S)$  is  $\sum_{v \in S, u \in V \setminus S} w(vu)$ . (Hint: Use (A) somehow.)

2. BOUNDED SET COVER.

[25 Points]

In the set cover problem, you are given an set  $X$  of  $n$  elements, and a family of  $m$  subsets  $\mathcal{F}$  of  $X$ . The task is to find a minimum number of sets of  $\mathcal{F}$  that covers all the elements of  $X$ . Let  $\text{opt}$  denote the number of sets in the optimal solution.

In the bounded set cover problem, you also know that every set has size at most  $u$ .

(A) Let  $\alpha_i$  be the number of elements covered for the first time in the  $i$ th iteration of the greedy algorithm. Prove that  $\alpha_i \leq u/2$ , for  $i > 2\text{opt}$ .

(B) Bound the number of sets output by the greedy algorithm as a function of  $u$ ,  $n$ ,  $\text{opt}$ , and  $m$  (your function might not depend on all these parameters). Your bound should be as low as possible (ignoring constants). Prove your bound.

3. NOT RANDOM WALK.

**[25 Points]**

Let  $S$  be a set of  $n$  elements  $S = \{s_1, \dots, s_n\}$ . At every round, you can randomly pick (with repetition) one element  $r_i$  of  $S$  (with all elements having probability  $1/n$  to be picked). For example, if  $S = \{1, 2, 3\}$  it might be that in the first round you picked the number 3, in the second round you picked the number 1, in the third round you picked the number 3 again, and in the fourth round you picked the number 2, and so on.

(A) **[10 Points]** Assume you already picked  $i$  distinct elements from  $S$  (let  $I$  denote this set of distinct elements), and let  $X_i$  be the number of additional number of rounds you perform till you pick a new element (i.e., an element that is in  $S \setminus I$ ).

What is the expectation of  $X_i$ ?

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(B) **[10 Points]** Starting with the empty set, in expectation, how many rounds one needs to pick all the  $n$  distinct elements of  $X$ ?

(C) **[5 Points]** Let  $t$  be the number of rounds one has to perform till one picked all the elements of  $S$  with probability larger than  $1 - 1/n^{10}$ . Give a bound on  $t$ , and prove your answer.

4. INTERVAL COVER.

**[25 Points]**

You are given  $X = \{1, 2, 3, \dots, n\}$ , and a set  $\mathcal{I} = \{I_1, \dots, I_n\}$  of  $n$  intervals on the real line. That is, for every interval  $I_j$  you are given two numbers  $a_j \in X$  and  $b_j \in X$  such that  $I_j = [a_j, b_j]$  covers all the numbers of  $X$  between  $a_j$  and  $b_j$ . Describe an algorithm that, in polynomial time, finds a subset of the intervals of  $\mathcal{I}$  such that they cover all the numbers of  $X$ , and this subset is as small as possible.

What is the running time of your algorithm? Provide a pseudo-code for your algorithm.