CS 273: Intro to Theory of Computation, Fall 2007
Problem Set 11 (due December 6th)

This homework contains 4 problems. It is due 12:30 in class or by noon at Elaine Wilson's office (3229 Siebel).

1. **Dovetailing**

   Fix an alphabet Σ, with $c \in \Sigma$. Let L be the set of all codes of Turing machines over Σ, which write a 'c' to the tape at some point for at least one input. For the purposes of this problem, reading a c from some tape cell without changing it does not count as "writing a c." Show that L is Turing-recognizable.

   That is, give a Turing machine which halts and accepts on an input w if and only if $w = \langle M \rangle$, where

   - M is a Turing machine, and
   - for some string $x \in \Sigma^*$, if you run M on input x, M writes c to the tape at some point.

   For example, if $w = \langle M \rangle$ for some M, where M is a Turing machine which moves its head to the right indefinitely on all inputs, then $w \notin L$. If $w = \langle N \rangle$, where N is a Turing machine which writes a c at the 100^{th} space on the tape if and only if the input is $cccc$, then $w \in L$.

2. **Yet another reduction**

   Show that the following language is undecidable.

   $$L_{prime} = \{\langle M \rangle \mid L(M) = \text{set of all prime numbers}\}$$

   Prove this using a reduction. Do not simply invoke Rice's Theorem. Although this would be a correct proof, the point of this problem is to learn how to write reductions.

   However, you should use the similarity with the Rice's Theorem proof to help you decide which language you should reduce from, and how the reduction could go.

3. **The proof of Rice's Theorem**

Let $L = \{\langle M \rangle \mid M$ is a Turing machine and $L(M)$ is finite$\}$.

Look at the proof of Rice's theorem in Sipser (problem 5.28 and its solution) and/or in Lecture 23. The two proofs are basically the same, except that Sipser's set P is S in the lecture notes, and the hypothetical decider for P is R_P in Sipser and N in the notes. Let's use Sipser's names for this set and this decider.

For this problem, you will write a proof that L is undecidable, following as closely as possible the proof of Rice's theorem.

(a) Clearly (but briefly) describe what's in the set $\overline{P}$.

(b) Is the set $T_\emptyset$ in P? Are you going to write your proof using P directly, or by switching to $\overline{P}$? (If you plan to switch to $\overline{P}$, answer (c) and (d) using $\overline{P}$ as your set.)

(c) Pick a suitable value for the Turing machine T. Describe $L(T)$ and, in no more than two sentences, how T works.

(d) The input to R_P is the encoding of a Turing machine $\langle M \rangle$. Explain briefly in English which inputs $\langle M \rangle$ accepts.

(e) Write out the proof that L is undecidable. Keep the structure of the proof the same as in the proof of Rice's theorem, substituting specific values (e.g. for the set P) as needed. (Minor variations in wording don't matter.)

4. **Applying Rice's Theorem**

For each of the problems below, determine whether the problem is undecidable as a direct corollary of Rice's theorem. If you think it is not, state precisely why Rice's Theorem cannot be applied. If you think it does follow from Rice's theorem, argue why, and especially argue why the non-triviality condition required for Rice's theorem holds. Keep your answers to the point and brief.

(a) L is the set of encodings of Turing machines that compute whether a number is even.

(b) L is the set of encodings of Turing machines that read any tape position at most 3 times.

(c) L is the set of encodings of Turing machines that accept *all* its inputs.

(d) L is the set of encodings of Turing machines that decide A_{TM} (where A_{TM} is defined as in Sipser).