CS 273: Intro to Theory of Computation, Fall 2007
Head-banging 1 (12-14 Sept)
Solution to problem 3

3. Structural induction

Here is a recursive definition of a set S containing points in the plane:
Rule 1: $(3, 0)$ is in S.
Rule 2: If $(x, y) \in S$ and $(p, q) \in S$, then $(x - p, y) \in S$.
Rule 3: If $(x, y) \in S$, then $(y, x) \in S$.
Rule 4: S is the smallest set satisfying rules 1-3.

Show formally that S is equal to the set $T = \{(x, y) | x, y \in \mathbb{Z}$ and x,y are multiples of 3$\}$

(a) Show that $S \subseteq T$.

(b) Show that $T \subseteq S$.

Solution. (a) We need to show that everything we can get through applications of the above rules (i.e. any element of S) is a pair of multiples of three. We shall prove this using structural induction.

BASE CASE: By rule 1, $(3, 0) \in S$. It is clear that both 3 and 0 are multiples of 3, so $(3, 0) \in T$ as well.

INDUCTION STEP: Assume that we have two pairs $(x, y), (p, q) \in S$ which are also in T, and therefore pairs of multiples of 3. Say $(x, y) = (3x', 3y')$, and $(p, q) = (3p', 3q')$. Then applying rule 2 to these pairs, we see that $(x-p, y) = (3x'-3p', 3y') = (3(x'-p', 3y') \in S$. Thus $(x-p, y) \in T$. Similarly, applying rule 3 to (x, y), we see that $(y, x) = (3y', 3x') \in S$. Thus (y, x) is also a pair of multiples of three, and hence $(y, x) \in T$.

Thus by induction, $S \subseteq T$. QED

(b) We need to show that we can construct any pair of multiples of 3, using only the rules 1-3. All you need for this is the regular induction you all know and love. I broke this problem up into a few parts.
Claim: For each $n \in \mathbb{N}_0$, $(-3n, 0) \in S$.
Proof: **BASE CASE:** $(n = 0)$ $(0, 0) \in S$, by applying rule 2 to the pairs $(3, 0)$ and $(3, 0)$.
INDUCTION STEP: Now assume that $(-3n, 0) \in S$. Then applying rule 2 to the pairs $(-3n, 0)$ and $(3, 0)$, we see that $(-3n-3, 0) = (-3(n+1), 0) \in S$. Thus the claim is true by induction. QED

For each $n \in \mathbb{N}$, we can apply rule 2 to the pairs $(0, 0)$ and $(-3n, 0)$ to show that $(3n, 0) \in S$, and apply rule 3 to $(\pm 3n, 0)$ to see that $(0, \pm 3n) \in S$. Now let $(3x, 3y) \in T$. Then applying rule 2 to the pairs $(3x, 0)$ and $(0, 3y)$ (which we know are in S by the above arguments), we see that $(3x, 3y) \in S$. QED