

Sample proof for set equality with recursive definition

Problem

Let set S be defined inductively over the integers as follows:

Rule 1: 7 is in S .

Rule 2: If $x \in S$ and $y \in S$, then $x - y \in S$.

Rule 3: No other elements are in S except those given by a finite number of applications of rules 1 and 2.

Claim: S is the set of all multiples of seven, i.e. all numbers of the form $7n$, where n is an integer.

Proof

Let T be the set of all multiples of seven. To show that $S = T$, we need to prove that $S \subseteq T$ and $T \subseteq S$. (This is typically the best outline for a proof that shows two sets are equal.)

To show that $T \subseteq S$, first notice that all members of T are of the form $7n$. Also notice that 7 is in S by rule 1 and 0 is in S by applying rule 2 with $x = y = 7$.

Now use induction on $|n|$ to show that all elements of T are in S .

Base: $n = 0$. Then $7n = 0$. We already know that zero is in S .

Induction: Suppose that $7n$ and $-7n$ are in S , for some $n \geq 0$. We need to show that $7(n+1)$ and $-7(n+1)$ are in S .

Since $-7n$ is in S (by the inductive hypothesis) and 7 is in S , by rule 2 tells us that $7 - (-7n)$ is in S . But $7 - (-7n) = 7 + 7n = 7(n+1)$. So $7(n+1)$ is in S . Since we know that zero is in S , then rule 2 also tells us that $0 - 7(n+1)$ is in S . I.e. $-7(n+1)$ is in S .

We now need to show that $S \subseteq T$. Use induction on the number of times a rule has been applied in creating each element of S .

Base: suppose that w is an element of S created using only one rule application. Then we created w by applying rule 1 once. So w must be 7, which is obviously in T .

Induction: suppose that every element of S created with no more than n rule applications is in T . We need to show that elements created using $n+1$ rule applications are also in T .

Let w be an element of S created using $n+1$ rule applications. Consider the last rule that we applied in creating w . If it was rule 1, then w is 7, which is clearly in T .

Now suppose that w was created by applying rule 2 to some elements x and y in S . x and y must have each been created using no more than n rule applications. So x and y are in T by our inductive hypothesis. So x and y are both multiples of seven. Therefore their difference w must also be a multiple of seven.