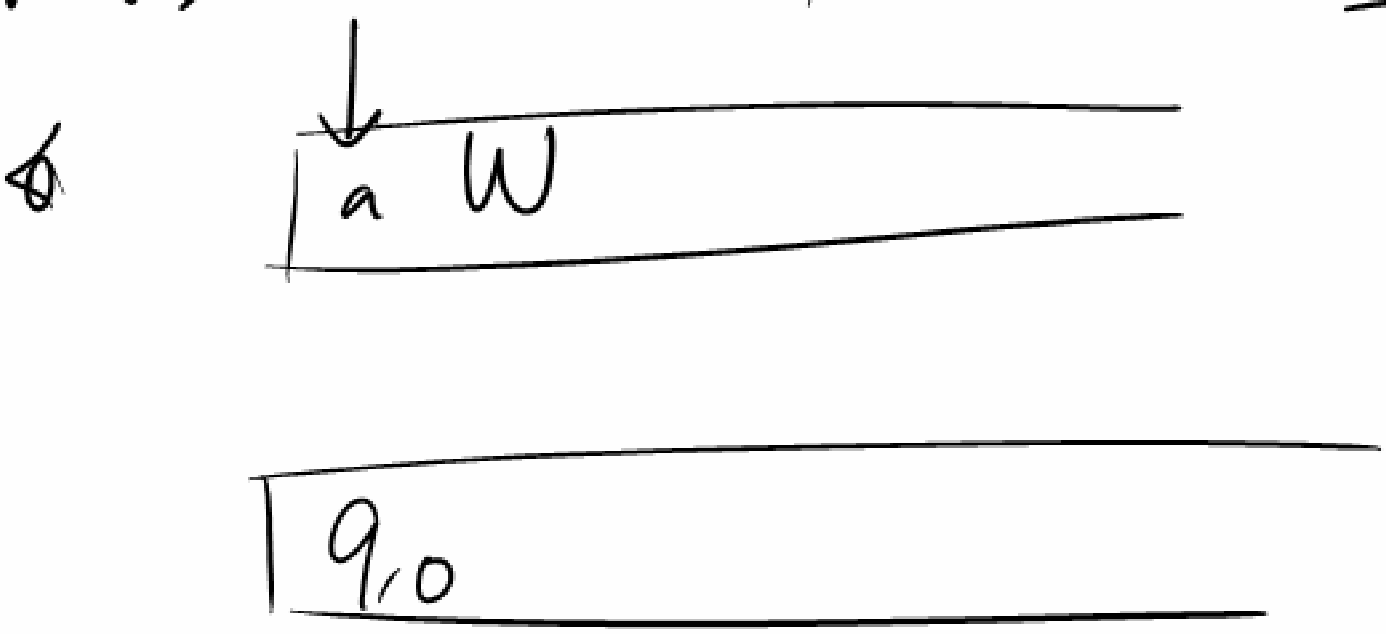
Note Title 11/7/2006

Undecidability

$$A_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$$

A_{TM} is Turing-recognizable.

$\langle M\rangle = Q, \Sigma, \Gamma, \underline{\underline{\delta}}, q_0, F$



accept if M accepts w

reject if M rejects w

not halt if M does not halt.

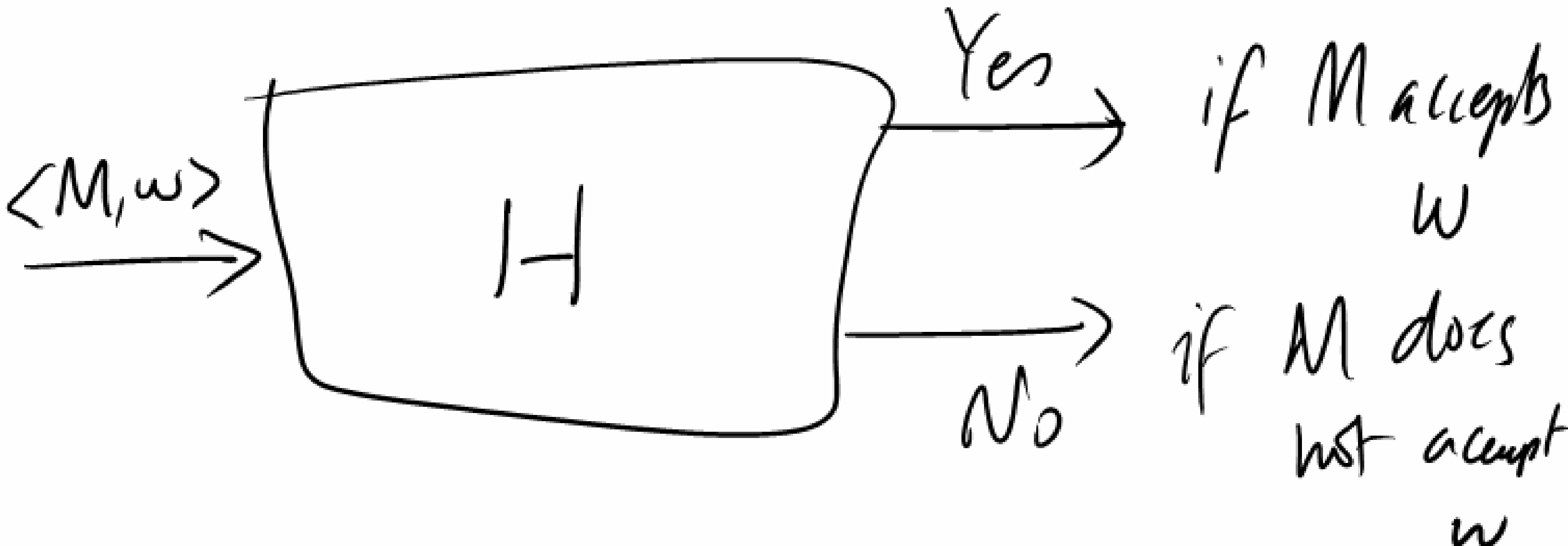
A_{TM} is <u>not</u> decidable.

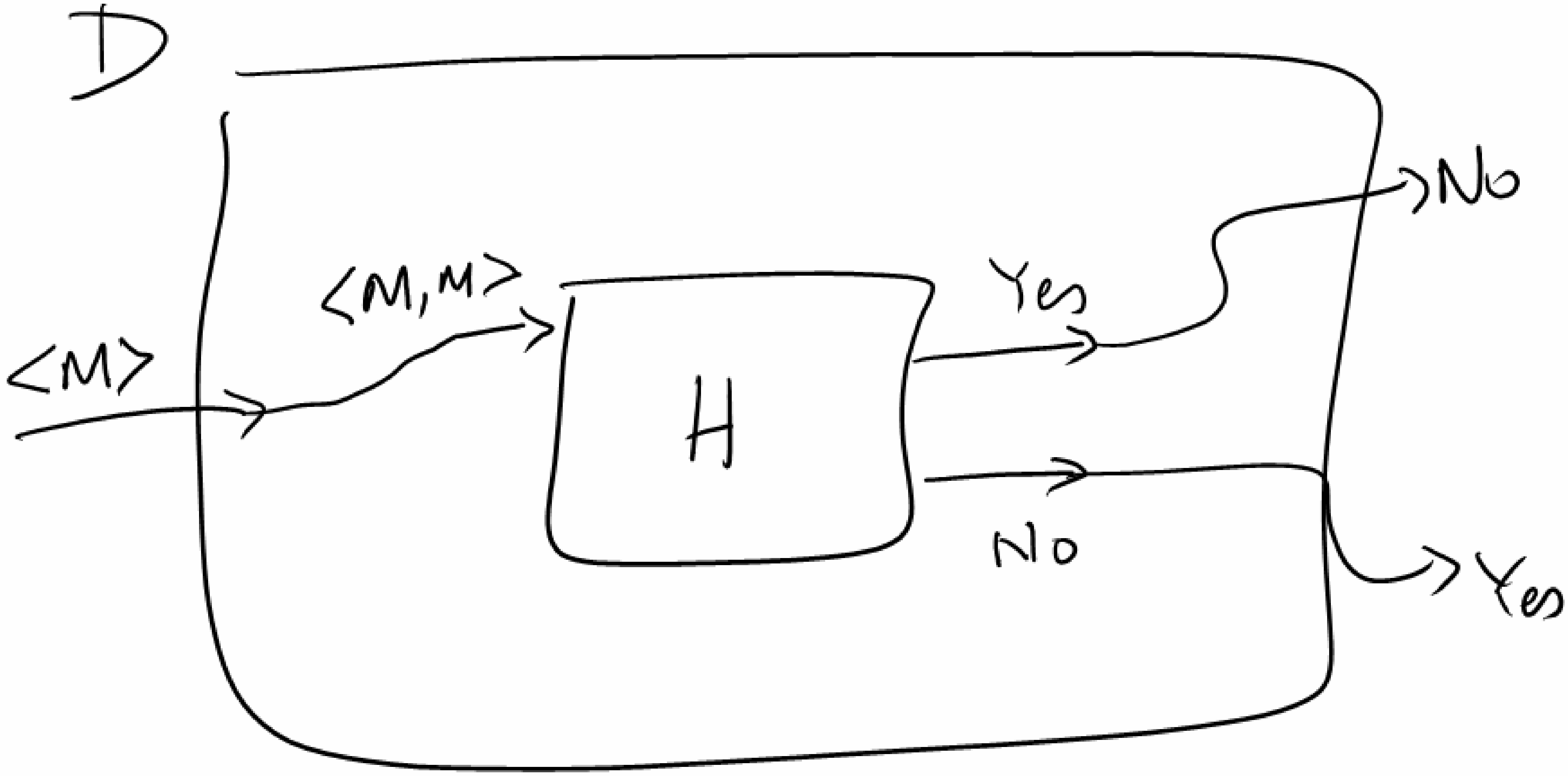
i.e. there is no Turing that halts on all inputs and accepts A_{TM}.

<u>Proof by contradiction</u>

Assume A_{TM} is decidable.

Let H be a TM that decides A_{TM}.





D is a decider? Yes

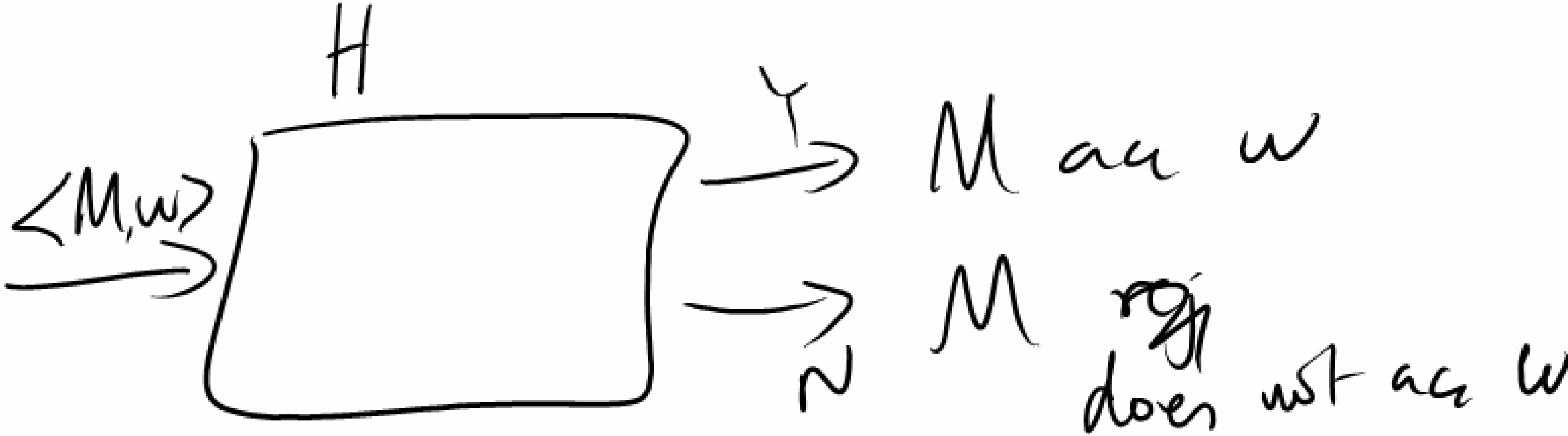
$D(\langle M \rangle)$ = accept if M does not accept $\langle M \rangle$

= reject if M accepts $\langle M \rangle$

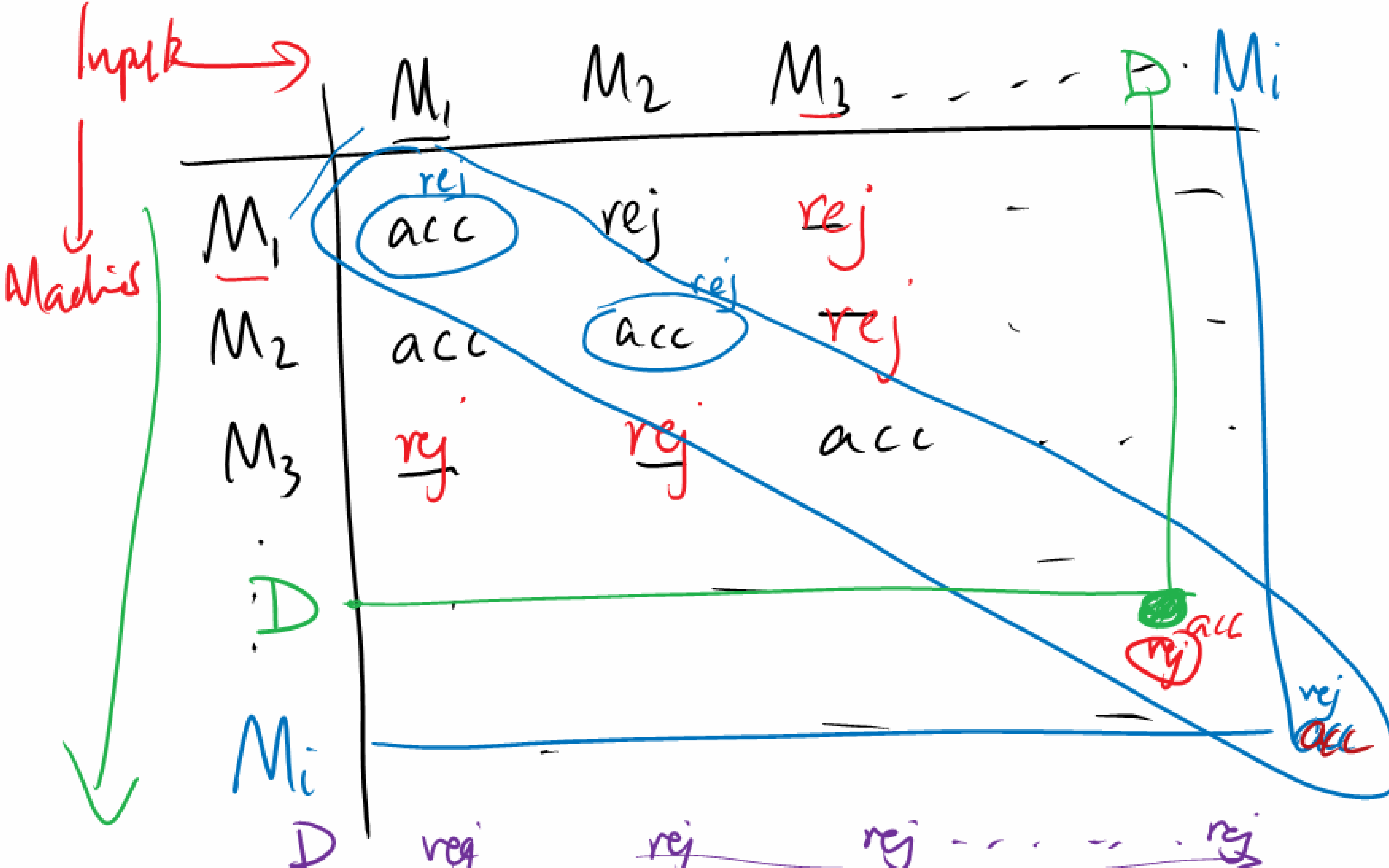
$D(\langle D \rangle)$ = accept if D does not accept $\langle D \rangle$

= reject if D accepts $\langle D \rangle$

Contradiction!



$\langle H, \langle H \rangle \rangle \longrightarrow H \longrightarrow$ Yes ✓



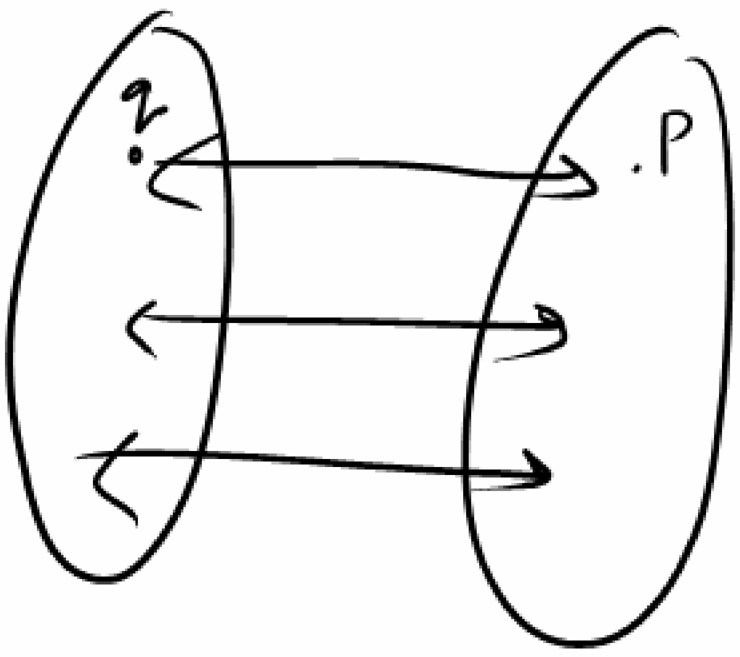
H – Completes the table

D – Diagonal language inverted

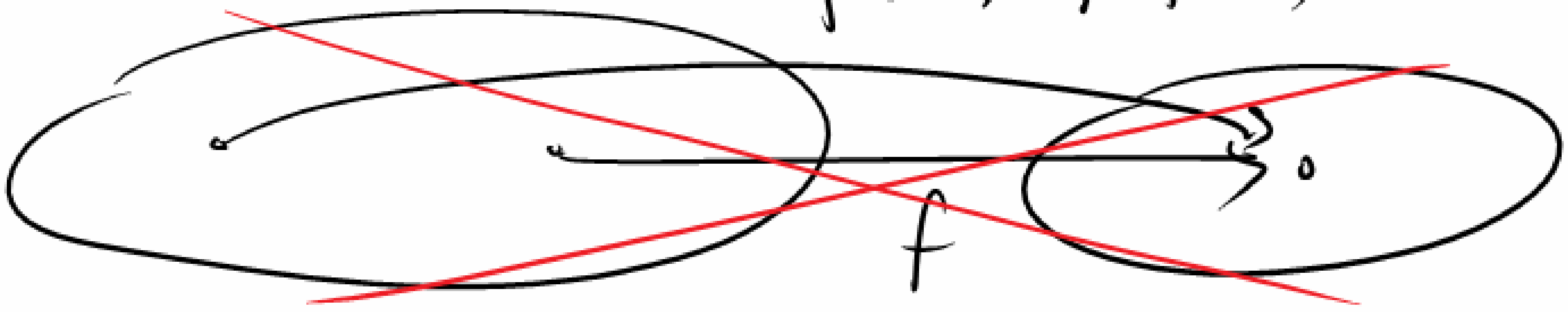
Cantor's universe

$$|A| = |B|$$

A & B are of the same size if $|A| = |B|$.



$f : A \to B$ is <u>one-to-one</u> if $\forall d, d' \in A$, $d \neq d'$, $f(d) \neq f(d')$.



~~4~~ f is <u>onto</u> if "f covers B" ie. $\forall e \in B \ \exists d \in A$ $f(d) = e$.

f is a <u>correspondence</u> between A & B if f is one-one and onto.

Eg $f: \{a, b, c\} \longrightarrow \{1, 2, 3\}$

$$f: \begin{aligned} a &\mapsto 1 \\ b &\mapsto 2 \\ c &\mapsto 3 \end{aligned}$$

is a correspondence.

Two sets A and B are of the same size if there is a correspondence between them.

$$A = \{1, 2, 3, \ldots\}$$

$$B = \{2, 3, 4, \ldots\}$$

$$f: A \to B \qquad f: i \mapsto i+1$$

$$A = \{1, 2, 3, \ldots\}$$ — countable

$$B = \{2, 4, 6, \ldots\}$$

$$f(i) = 2i$$

A set A is <u>countable</u>

if A is of the same size as $\mathbb{N}$.

$$A = \{ a_1 , a_2 , \ldots \}$$

$$\mathbb{N} \to \{ 1, 2, 3, \ldots \}$$

$$\text{Rationals} = \{ a/b \mid a, b \in \mathbb{N} \}$$

	1	2	3	4	…
1	1/1	2/1	3/1		
2	1/2	~~2/2~~	3/3		
3	1/3	.	.		
4					
⋮					

Traverse the table, omit repeated rationals.

The set of rationals is countable.

$\mathbb{R} =: 237.357929\ldots\ldots$

$\mathbb{R}$ is not countable!

Assume R is countable

& $f : \mathbb{N} \to R$ is a correspondence

n	$f(n)$
1	$3.15792\ldots$
2	$2.64952\ldots$
3	$0.22795\ldots$
$\vdots$	
i	$5.792\ldots$

$\neq$ $\neq$ $\neq$

7

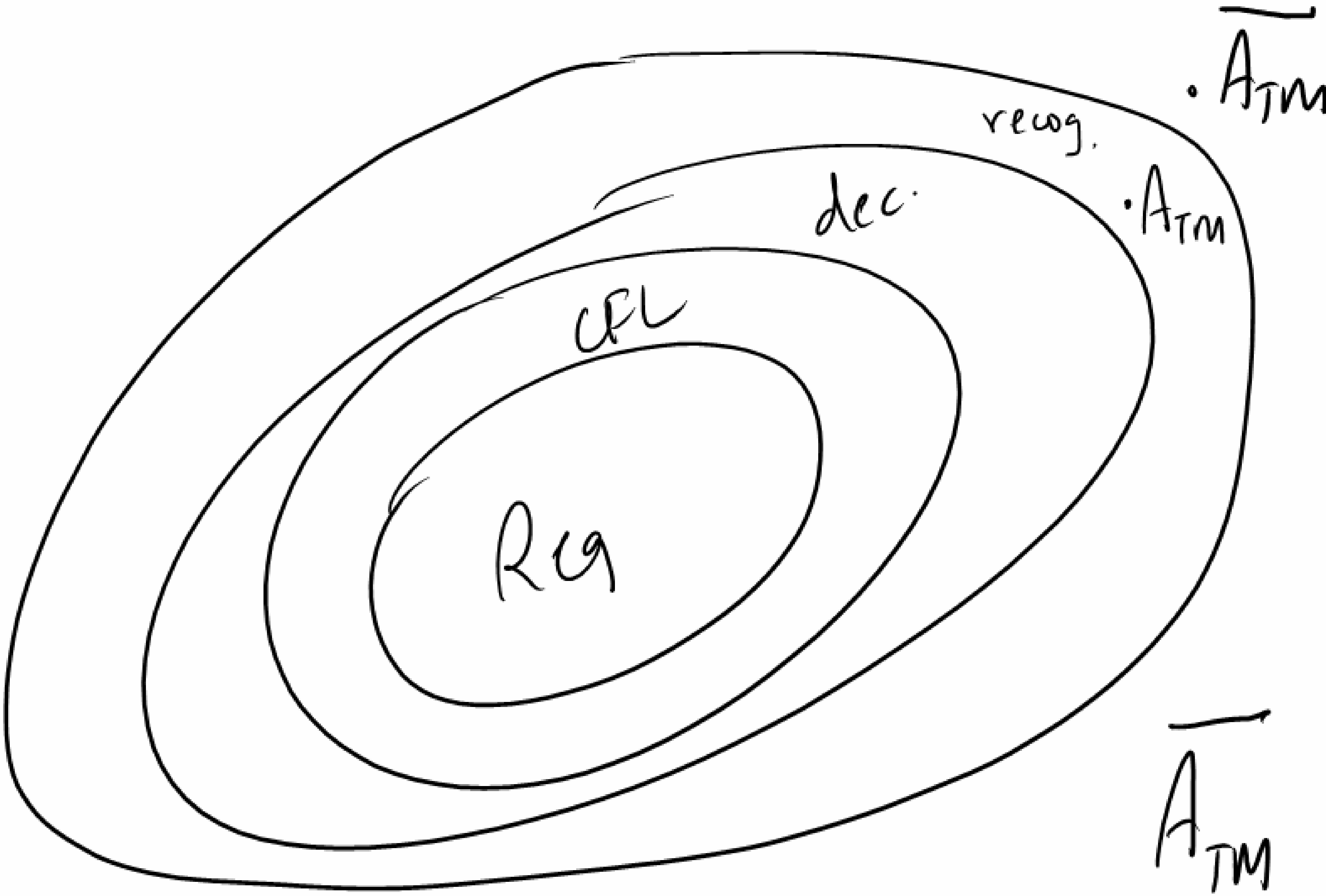
r

i^{th} digit

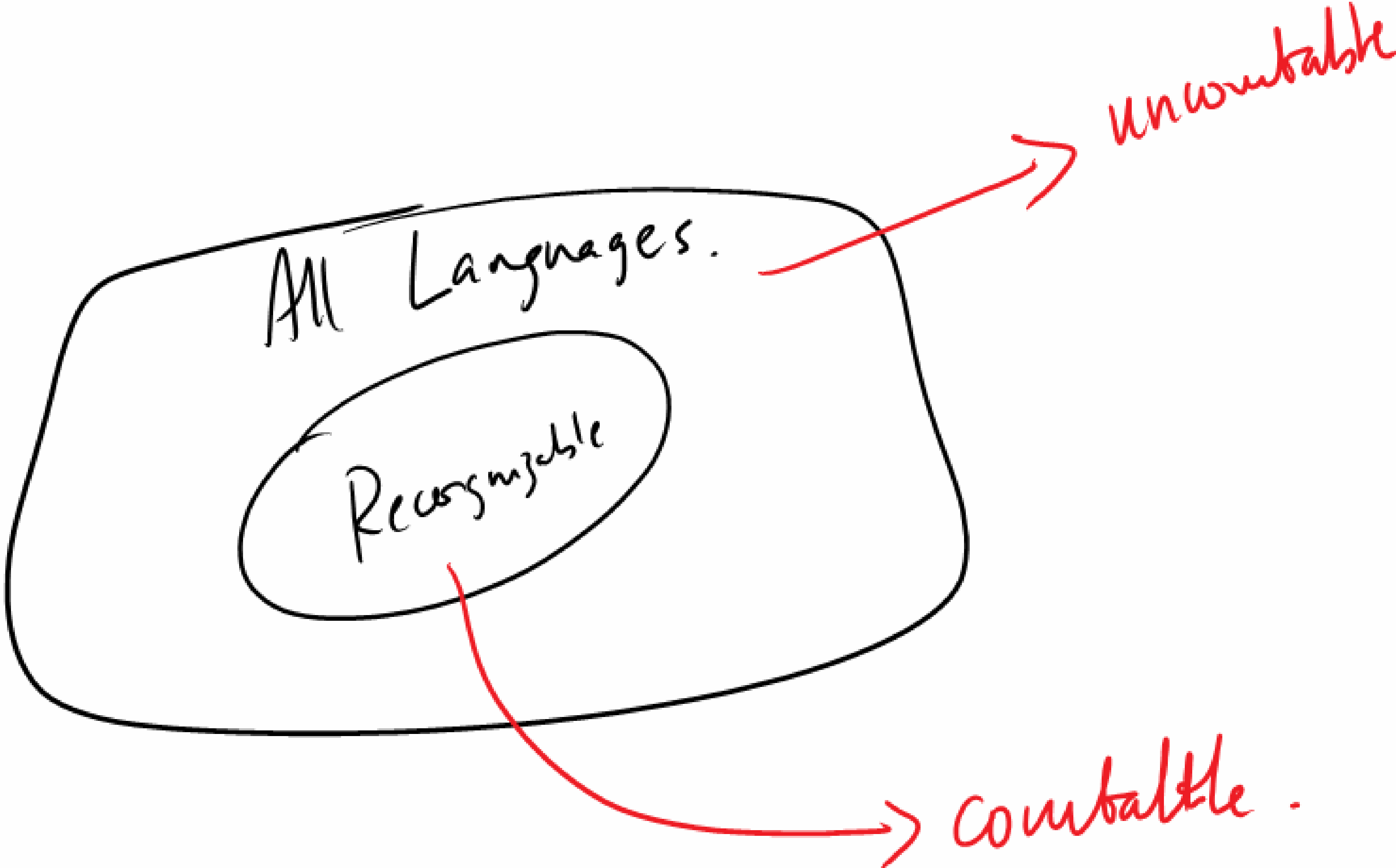
$r = 4.85\ldots7$

$\uparrow$ i

So R is <u>not</u> countable.



$\overline{A_{TM}}$ is <u>not</u> Turing recognizable.


All Languages.
Recognizable
uncountable
countable.

The class of recognizable languages

is <u>countable</u>.

recogn $\longleftrightarrow$ M ← Turing machine.

$$\Sigma^* = \{ \epsilon, \ 0, \ 1, \ 01, \ 10, \ 001, \ \ldots \}$$

$$TM = \{ \underline{\underline{M_1}}, M_2, M_3, M_4 \ldots \}$$

The set of all TMs accepting recognizable languages is <u>enumerable</u>.

<u>Lemma</u> The set of infinite strings over $\{0,1\}$ is uncountable.

n	$f(n)$
1	0101001110 ------
2	100010101111 .....
3	10100111 ..
4	1011 .....

1100 ....

The set of <u>all</u> languages over Σ is <u>uncountable</u>.

$$\Sigma^* = \{ \varepsilon, \ 0, \ 1, \ 01, \ 10, \ 11, \ \ldots \}$$

$$L: \ \checkmark \quad \times \quad \checkmark \quad \times \quad \times \quad \checkmark \quad \ldots$$

$$enc(L): \ 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad \ldots$$

The class of all languages is in correspondence with the set of infinite strings over $\{0,1\}$.

So class of all languages is uncountable

So Recognizable lang $\neq$ All languages.